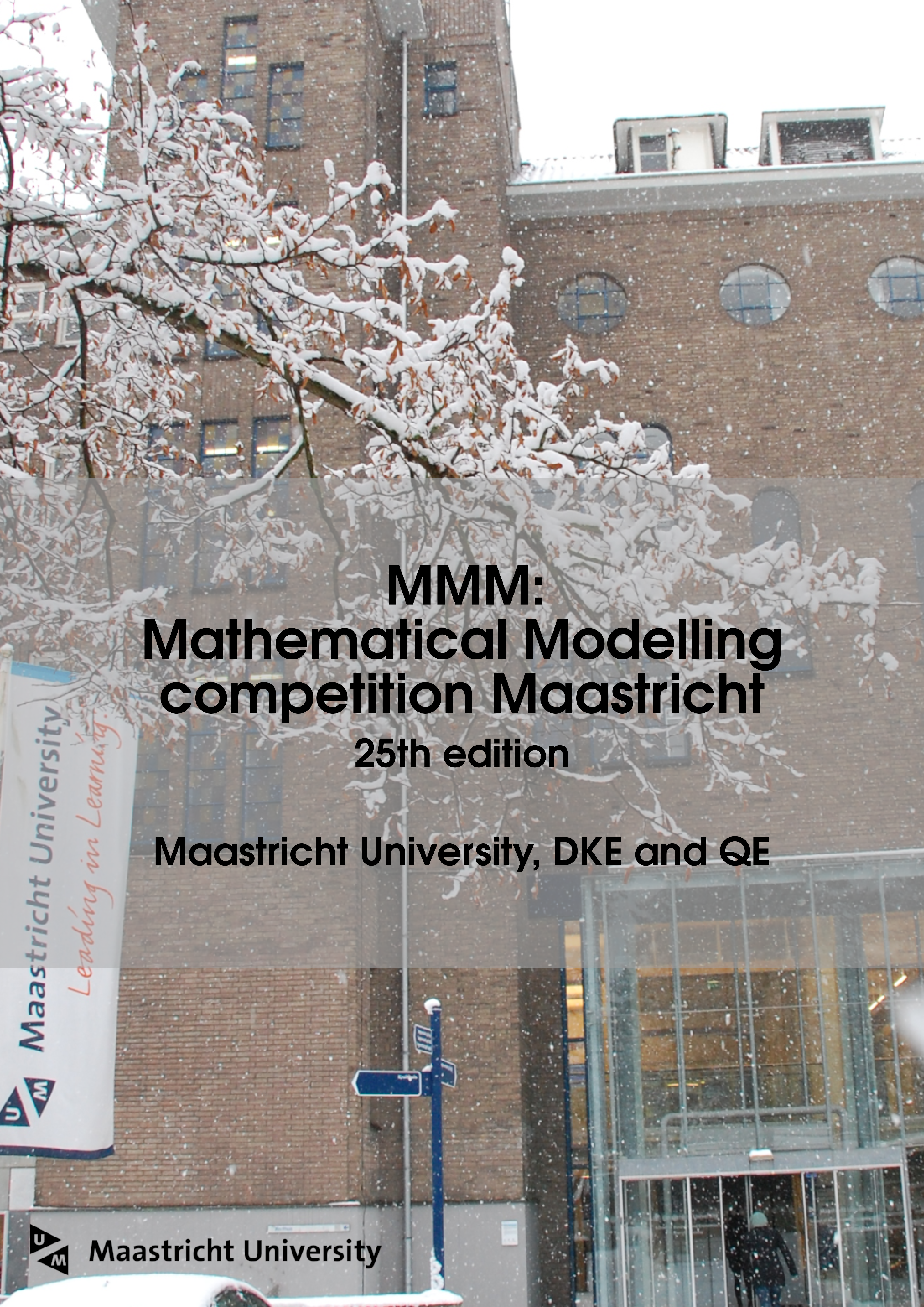

MMM: Mathematical Modelling competition Maastricht

25th edition

Maastricht University, DKE and QE

Maastricht University

PROJECT.DKE.MAASTRICHTUNIVERSITY.NL/MMM/

On Saturday January 26, 2019, the special 25th edition of the 'Mathematical Modelling Maastricht' (MMM) competition will take place. This MMM-competition started in 1995 as a small-scale initiative to draw attention to two new study programs at Maastricht University: the current programs *Data Science and Knowledge Engineering* and *Econometrics & Operations Research*. It has developed into an yearly independent yearly event with 200 participants (high school students) from Belgium, Germany, and the Netherlands. On the occasion of this 25th edition this booklet is published with information about the past 25 years MMM. It also contains the problem set of the 25th edition (pages 33 - 35).

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Contents

$2^{2019} - 2^{2018} = ?$

1. Preface: aim and background

In 1992 and 1993 two new study programs started at Maastricht University. *Knowledge Engineering* (Kennistechnologie) was a joint enterprise of the Department of Mathematics and Computer Science of the Faculty of General Sciences of the UM and the Limburgs Universitair Centrum in Diepenbeek, Belgium, which was affiliated with the University of Leuven. The focus of this program was on operations research and computer science. *Econometrics and Operations Research* was a new program within the Faculty of Economics and Business of Maastricht University, offered mainly by the Department of Quantitative Economics. The focus of this program was on econometrics, mathematical economics, and operations research. These were the first programs at Maastricht University of a strong mathematical nature and together with the Maastricht Science Program they are, in fact, still the only programs at the UM which can be called exact according to Dutch terminology.

At that time, the interest in mathematical or applied mathematical study programs in the Netherlands was not overwhelming and, moreover Maastricht University was not exactly known as a place which offered such programs, let alone had much of a reputation in this respect. Not surprisingly, the inflow of students in the two new programs at the beginning was quite low. The Mathematical Modelling Maastricht (MMM) competition for high school students was one of the initiatives taken which intended to increase visibility and to boost the student numbers for both programs.

The set-up of the MMM aimed to reflect two common characteristics of the two study programs: problem based learning and mathematical modelling. The former is reflected by the fact that from the beginning on participants in the MMM cooperate in small groups. The latter appears from the fact that many of the MMM problems require students to first set up a mathematical model for the problem at hand: in fact, the problems often look like so-called ‘redactiesommen’ from earlier times.

The MMM started on May 13, 1995, with about 50 participants. The next edition took place in January 1996, and after that always in January, which was more suitable than Spring, in view of final high school examinations. After a year with 80 participating groups and in total 400 participating students, registrations for the MMM were limited to 40 groups and 200 students.

Meanwhile, names have changed: the MMM is now organized by the Department of Data Science and Knowledge Engineering of the Faculty of Science and Engineering and the Department of Quantitative Economics of the School of Business and Economics. The two study programs *Data Science and Knowledge Engineering* and *Econometrics and Operations Research* attract much larger numbers of students than 25 years ago: up to 300 students in total. One of the objectives of the current MMM still is to draw attention to these two programs, but the marketing aspect has been toned down: the chief objective is the sheer fun of the event, both for the organizers and the participants. No doubt, also this special 25th edition of the MMM will be an interesting and cheerful event.

Hans Peters
Gijs Schoenmakers

We thank everyone who contributed to the MMM competitions in the past 25 years.

This booklet was composed by Karin van den Boorn, Jean Derks, Ralf Peeters, Hans Peters and Gijs Schoenmakers.

A duck swims in the middle of a circular pond with a menacing cat outside. The cat runs four times as fast as the duck can swim, but cannot enter the water. Can the duck get to the border of the pond without the cat being on top of him?

2. Anecdotes

2.1 'Is this going to be cool?'

About twenty years ago I participated in MMM. My math teacher had asked me to join and I remember I wasn't immediately enthusiastic. It would be on a Saturday, a day I normally worked. And I thought it wasn't a very cool thing to do on a weekend. Boy, was I wrong! My teacher still had a place vacant in the girls team. 'You are a good addition' he encouraged me. The smartest boys of my year had already formed a team. First place and nothing less was their motto, which they made public to everyone in school. My concerns were rather 'Can I actually solve any of those questions?' and 'Is this going to be cool?'. The relaxed attitude of the girls – 'Just join! It will be fun' – convinced me to join.

I liked math. I was good at it. Obviously something my math teacher knew. At the time I wasn't that serious about what to do after my last year in high school. Studying Econometrics had crossed my mind, but it could just as well be something else. I knew Maastricht university offered an Econometrics program and also something which was called 'Problem based learning'. Whatever that may be, it sounded interesting. I signed up for MMM because it would be a nice day with my friends, but also a little because I was curious about Maastricht university.

So on a Saturday morning really early, we all gathered at the train station. During the ride the girls were chatting, excited about the day out. The boys were checking exercises from an earlier edition of MMM. The contrast couldn't have been bigger and thinking back about it, I felt a little disappointed that they were so focused on winning.

When we got to the university, our math teachers were already waiting there for us. They were exited too: some interesting lectures on math, while we were working on the assignments. After the introduction in the aula and encouragements from our teachers it was time to start. Every team was assigned a room number. Our first hurdle: find our room. After some running back and forth we finally got there. Immediately bags were turned inside out to gather all snacks and drinks for the day. Then we divided the assignments, two people per exercise. The fun part started: We were brainstorming out loud in a corner of the room, using the blackboard, using paper. I think I was responsible for solving the question ‘How many colors do you need for a map of the world when no neighboring countries can have the same color?’. In the last half hour all answers were put together and as a team we tried to solve the last remaining questions.

> The exercises were really interesting and the way we solved them really inspired and motivated everyone in the team. It felt really good to work like this.

And after we handed in our answers we were all excited: we had done this together!

Then we had to wait for the results. Our team surprisingly ended somewhere top of the middle. Not far below the boys team. Against their own expectations, the first place was apparently not theirs. An anticlimax for them. Luckily their sadness lasted shortly. Our enthusiasm about the day made for a nice train ride back. And plans were made for a night out together, boys and girls.

And what happened with me? I went to Maastricht to study Econometrics, followed by a PhD in game theory. MMM has had some influence on this decision. The way we tackled problems as a team that day felt really good. The small scale of Econometrics in Maastricht at the time I studied there, and working in small teams on assignments: often, studying felt like that day at MMM.

Caroline Berden, Nederlandse Zorgautoriteit

2.2 ‘Vallen we toch buiten de prijzen?’

Het is zaterdag 26 januari 2013 en ik rijd terug naar Schoten (België) met vijf leerlingen. Ze hebben die dag deelgenomen aan de wiskundecompetitie in Maastricht. Het resultaat is niet goed: ze eindigden niet als laatste, maar het scheelde niet veel. Is de stemming in de auto bedrukt? Helemaal niet! De teneur: ‘Volgend jaar komen we terug!’ Het zijn dat jaar inderdaad allemaal leerlingen van het voorlaatste jaar van het secundair onderwijs die deelnemen. Ze krijgen het jaar daarna dus nog een kans. En of ze die grijpen!

Tijdens de volgende editie van de Mathematische Modelleercompetitie Maastricht zit de sfeer van in het begin goed. Er wordt zelfs gezongen in de auto! De wedstrijd verloopt naar wens: ze voelen dat het goed ging maar hebben over sommige vragen toch nog wat twijfels. Tijdens de voorstelling van de oplossingen krijgt de ploeg veel bevestiging over de kwaliteit van hun antwoorden. De proclamatie is spannend, want de verwachtingen zijn hoog. Wanneer ze niet uit de bus komen bij de bekendmaking van de derde plaats, is er twijfel tussen ‘Vallen we toch buiten de prijzen?’ en ‘Scoren we nog hoger?’ Als ook de tweede plaats niet voor hen is, wordt dat gevoel alleen maar sterker. Het duurt niet lang voor de onzekerheid weggenomen wordt: ze zijn eerst! De rest van de dag verloopt in triomf! Het hierboven geschetste verloop van de wedstrijd is atypisch. Meestal neemt onze school deel met een ploeg van laatstejaars, die bijgevolg maar één keer kunnen inschrijven. Het verhaal begint in oktober, bij het voorstellen van de competitie in de klas(sen), het aanbieden van een smaakmaker voor de vraagstelling

The 1st prize winning team from Schoten, in 2014

en het enthousiasmeren tot deelname. Als het lukt om een ploeg samen te stellen, komen we enkele keren samen om wat te oefenen en elkaars sterke en zwakke kanten te leren kennen. Mijn raad luidt steevast: 'Wees kritisch voor elkaar maar maak zeker geen ruzie!'

> The results are unpredictable, but that is not a problem. The Maastricht competition is a nice opportunity to step outside the restricted context of the school, to get acquainted with totally different mathematical applications (in another language), and to sharpen the ambition for extensive study.

De resultaten wisselen heel sterk van jaar tot jaar, maar dat is niet erg. De Maastrichtse competitie vormt een goede gelegenheid om eens buiten de beperkte wereld van de eigen school te treden, kennis te maken met totaal andere wiskundetoepassingen (in een andere taal) en de ambitie om hard te studeren wat aan te scherpen. Dat de Universiteit Maastricht ons die kans biedt, is ondertussen goed bekend in onze school en resulteerde vorig jaar in de eerste oud-leerling die besliste om aan de UM te gaan studeren. Wellicht volgen er nog.

Dirk Kerstens, Sint-Michielscollege van Schoten

2.3 'An enjoyable experience, despite meagre results'

In my last year of high school my math teacher selected the top 5 pupils of our advanced course in mathematics to attend the competition in Maastricht. We were excited for the trip, which seemed like an excursion at the time. Later, as a student, I came to think of the 1.5-hour drive as a relatively short distance. Once we arrived, we participated in the problem solving competition, which posed challenging tasks and required team work in a way we were not used to at school.

It was an enjoyable experience, despite meagre results. The competition was followed by lectures of several faculties, in which I realised that the book I had been reading the night before (the German translation of 'Artificial Intelligence: A Modern Approach') was a core textbook used at DKE. We ended the day getting lost in Maastricht's beautiful back alleys, and when I left I was set to return for the next open day. I visited several open days, and started studying BSc Knowledge Engineering and Computer Science in 2005, graduated MSc in Artificial Intelligence in 2008 and received my PhD in 2012. Without this competition I might have missed this opportunity that lay just across the border.

Michael Kaisers, CWI Amsterdam

2.4 '4 gemotiveerde vwo5-leerlingen'

Ik kan me nog goed herinneren dat we de eerste keer als school deelnamen aan de MMM competitie. Met 4 gemotiveerde vwo5-leerlingen ben ik naar Maastricht gereden. Hartelijk ontvangen, zoals altijd, door enkele dames van de organisatie. Wij vonden het allemaal een beetje spannend wat er ging gebeuren. In die tijd waren er nog lekkere broodjes voor alle deelnemers, voordat de wedstrijd van start ging. Na de gemeenschappelijk instructie gingen mijn leerlingen aan de slag, ik volgde de interessante lezingen. Vol verwachting wachtte ik mijn leerlingen vervolgens op, erg benieuwd naar hun ervaringen. Enigszins teleurgesteld vertelden ze me dat ze geen enkele opgave hadden ingeleverd. Ze hadden het idee dat ze geen enkele einduitkomst hadden...... toen later bleek dat ze ook punten kregen voor tussenstappen, konden ze zich wel voor hun hoofd slaan.

> Rather dissappointed my students told my that none of the exercises were handed in: no feasible answers were found... they were about to kick themselves after being told that intermediate steps and modeling are counted as well!

We hebben ons toen voorgenomen om terug te komen en een jaar later te laten zien wat we waard waren! En dat is gelukt, een mooi vierde plaats was een prestatie om mee voor de dag te kunnen komen. Voor onze school de start van een mooie traditie om jaarlijks met een goed team naar Maastricht te gaan. Dank voor deze mooie kans!

Jörgen Moonen, Van Maerlantlyceum Eindhoven

2.5 Maastricht MMM – the hardest mathematics-competition ever?

Why this competition is quite hard? Let me explain...
For the students it is

- hard to get up early in the morning at Saturday;
- very hard to find the right room;
- quite harder to solve the really difficult problems;
- between hard and frustrating to see how easy the solutions are while they are explained in the presentations of Hans Peters and Gijs Schoemakers.

For the teachers it is

- hard to understand the lectures longer than the first two minutes;
- hard to arrive in time:
  - in 2015 it snowed in the night, the motorway was covered by snow, maximum speed approximately 60 km per hour;
  - in 2016 we had a breakdown of my car near Aachen. We spend 5 hour at a gas station waiting for the mechanic and returned to Bonn after that.
    While drinking a hot chocolate in the gas station, the owner asked the students: 'It seems you are smart in maths, could you please explain me a calculation...' They could and the day was not useless!

Every year my students enjoy the MMM-Competition, because

- the problems are diversified;
- they don't have to wait for the results, the prize ceremony takes place on the same day;
- the building, the mood, the drinks, the other students, ...

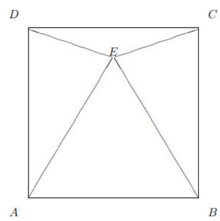
Thank you for the organization, I hope there will be further 25 year of MMM-competition in Maastricht.

Felix Hartenstein, Amos-Comenius-Gymnasium Bonn

2.6 Immer wieder ein Highlight

Bei uns am Suitbertus Gymnasium in Düsseldorf fiebern die besten Mathematiker jedes Jahr auf die Teilnahme am mathematischen Modellierungswettbewerb in Maastricht hin. Endlich mal ein Wettbewerb, bei dem sich nicht die Sportler sondern mathematisch begabte Schüler in einem Teamwettkampf miteinander messen können.

Unsere Schule erhielt erstmals eine Einladung zum achten Wettbewerb im Jahr 2002. Von Eltern begleitet fuhren wir nach Maastricht, ohne so richtig zu wissen, worauf wir uns einließen. Interessante Aufgaben, wie eine Langspielplatte der Beatles, die Türmchen von Hanoi oder gleichstarke Fußballteams, zeigten deutlich, wie vielfältig die Mathematik anwendbar ist. Mit erhitzten Köpfen kamen unsere fünf Teilnehmer nach zweieinhalb Stunden aus dem Seminarraum und diskutierten bei einer Erfrischung heftig über die gestellten Probleme. Bei der Besprechung der Aufgaben stieg die Spannung bis ins Unermessliche. Die Freude war riesengroß als klar war, dass wir den dritten Platz belegt hatten. Voller Stolz fuhren wir nach Hause.



Problem 1: Compute the angle

Consider the following diagram:

$ABCD$ is a square and $\angle CDE = \angle DCE = 15°$. Determine $\angle EAB$. Give a geometric argument, not using trigonometric concepts like sin, cos, tan, etc.

Getoppt wurde dieses Ergebnis drei Jahre später: Wir konnten als erste deutsche Schule den Wettbewerb gewinnen. Nach der Pisapleite endlich ein Lichtblick für unsere Mathematiker. Mir ist aus diesem Jahr eine Aufgabe besonders in Erinnerung geblieben, weil hier deutlich wird, wie ein zunächst kaum lösbares Problem durch eine geniale Idee so einfach wird, dass selbst Siebtklässler die Aufgabe lösen können. Johanna erinnert sich auch noch genau: ‘Malte gab uns den Tipp: Zeichne ein zum Dreieck CDE kongruentes Dreieck so in das Quadrat, dass die Seite CD auf die Seite DA fällt. Mit diesem Hilfsdreieck war die Lösung kinderleicht.’ So bleiben uns nur die besten Erinnerungen an die Fahrten zum MMM-Wettbewerb und bei Ehemaligentreffen schwärmen die Teilnehmer immer noch in den höchsten Tönen von den aufregenden Tagen in Maastricht.

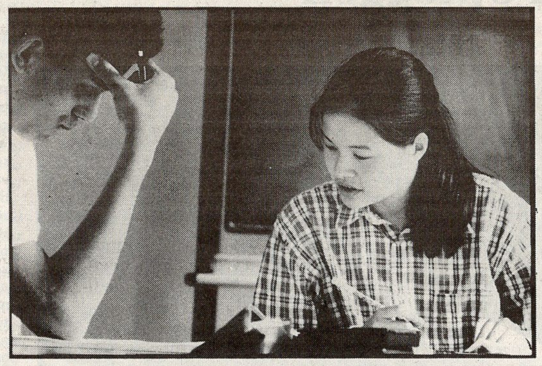
Vielen Dank an alle Organisatoren des einmaligen Wettbewerbs.

H. Kalenberg, Suitbertus Gymnasium Düsseldorf

„Weet u misschien wat boter, kaas en eieren is?”

Wiskundewedstrijd Nederland - België: 1 - 0

Vijfdeklassers mochten afgelopen zaterdag hun tanden stukbijten op wiskundige problemen Twee faculteiten, algemene wetenschappen en economie, organiseerden voor hen de eerste Mathematische Modelleercompetitie Maastricht



Sommetjes maken voor de echt (foto Nelis Tutkey)

Niet zomaar voor de flauwekul natuurlijk. De wedstrijd moet scholieren bekend maken met de exacte studies aan de Rijksuniversiteit Limburg: econometrie en kennistechnologie. Beide kunnen wel wat extra studenten gebruiken; kennistechnologie heeft een eerste lichting studenten, nu derdejaars, van slechts vier personen en ook de jaren daarna is het bij een handjevol studenten gebleven. Econometrie kan op gemiddeld vijfentwintig studenten rekenen, maar een stijging van het aantal aanmeldingen is meer dan welkom.Ook de andere kant van de medaille is interessant. De docenten krijgen op deze manier geheel gratis en vrijblijvend een aardig inzicht in het kennispeil van wat wellicht aanstaande studenten zijn.Van de vijftig deelnemers zijn er veertig Vlamingen. Die krijgen iets anders voorgeschoteld dan zij gewoon zijn, want het Belgische wiskunde-onderwijs is veel abstracter en theoretischer dan in Nederland, waar op de middelbare school meer aandacht wordt besteed aan toegepaste wiskunde. Een Belgische docent wiskunde, die een aantal van zijn pupillen vergezelt, vindt dat juist de toepassingen geen probleem zijn als men de theorie beheerst. Eén van de organisatoren van deze competitie, dr. Hans Peters, verzucht dat het Nederlandse wiskunde-onderwijs misschien wel te veel is doorgeslagen naar de toepassingen. “Dan wordt wiskunde net zoiets als een kookrecept.”

De groepjes scholieren zitten in aparte kamertjes, waar ze zich groepsgewijs over de vraagstukken buigen. De Nederlandse deelnemers in het eerste kamertje hebben de deur angstvallig op slot gedraaid. We worden pas binnengelaten nadat de camera van de fotograaf is goedgekeurd. Een knulletje verklaart zijn achterdocht door te zeggen: “Soms zijn we aan het werk en dan komt iemand binnen met zo’n compact-camera; nou dat is niks.” Als deze ster-allures een enerverende wiskundige discussie beloven, dan is dat een teleurstelling, want in dit lokaal overheerst het grote zwijgen. Het werk is verdeeld en ieder voor zich knobbelt op een opgave.

De volgende kamer wordt bezet door zes Belgen en de sfeer is aanmerkelijk chaotischer: een hoop bedrijvigheid en veel kabaal. De twee dames in het gezelschap turen op een horloge en worstelen met de ‘verwisselbare wijzerstanden’ van een klok. Drie heren hebben op het bord een spoorwegnetwerk uitgetekend. Ze proberen het maximale aantal treinen te berekenen dat gedurende één nacht een bepaalde route kan afleggen. Maar in hoog tempo wordt van het ene vraagstuk naar het andere ‘gezapped’. De Belgen hebben enigszins te kampen met een afwijkende woordenschat en opgave vijf blijkt dan ook niet te begrijpen. Totdat één jongen zich omdraait en vraagt: “Weet u misschien wat ‘boter, kaas en eieren’

Observant, May 18, 1995

Observant 34 • 18 mei 1995 6

is?” Eenmaal op het bord getekend is het spelletje welbekend, maar in België heet dat toch echt ‘tic tac tone’. Terug naar het spoorwegnetwerk. Eén van hen moppert: “Dat ze me maar vragen om een ‘integraal’, of iets anders dat ik direct op kan lossen, maar dit?”

Vooropgesteld dat tachtig procent van de deelnemers uit Vlaanderen kwam, was het aannemelijk dat een Belgisch team met de hoofdprijs ervandoor zou gaan. Toch won de Eindhovense school: het team met de deur op slot en de arrogantie van een verzameling grootmeesters. Vond de Belgische apologeet van de abstracte wiskunde in Maastricht zijn Waterloo?

Hoe dan ook, anders dan de bedoeling was zullen noch kennistechnologie, noch econometrie veel wijzer worden van deze wiskundewedstrijd voor vijfdeklassers. Tenminste, niet in de zin dat die volgend jaar aan de poort zullen staan om zich als student in te schrijven. Waarom niet? Omdat de deelnemers geen vijfdeklassers maar hoofdzakelijk zesdeklassers bleken. En die hebben hun studiekeuze al lang en breed gemaakt. Maar, zo besloot de voorlichter van kennistechnologie, het was toch een geslaagde middag geweest.

Marcia Luyten

2.7 Observant: MMM ‘not just for fun’

Right after the first edition of the MMM-competition, May 13, 1995, the Maastricht University newspaper *Observant* reported on the initiative behind MMM: the associated study programs ’may use some extra students’. Further, it provides the university teachers with extra information on the talents of the prospective students ‘at no costs, and without any obligation’. The heading refers to the fact that a Dutch team won the competition while Dutch and (many more) Belgian teams participated. *Observant* noted that the exercises were set up in typical Dutch (i.e., non-Flemish) vocabulary…

Een pak hagelslag, dat valt de Belgen zwaar

MAASTRICHT • Rekenen zullen ze wel kunnen, hoogleraren en doctores in de wiskunde, maar gewoon kijken bij welk getal rechts op een vel papier links het nummer van een deelnemende school hoort?

Het was een vermakelijke vergissing zaterdag in de aula van de Universiteit Maastricht. De tweede prijs van een wedstrijd wiskunde voor scholieren ging aanvankelijk naar team 17, van het Sint-Janscollege in Hoensbroek. Maar gelukkig kreeg een ander team van Hoensbroek de prijs. Dat stond een regeltje hoger en had bijna het dubbele aantal punten gescoord, 37 van de in totaal vijftig.
De wiskundewedstrijd werd voor de derde keer gehouden. Er deden dertien teams uit Vlaanderen mee en twaalf uit Groot Zuid-Nederland (met Nijmegen en Arnhem erbij). Winnaar werd hetzelfde team als vorig jaar: een meisje en twee jongens van het Gelders College uit Arnhem. Behalve Hoensbroek deden ook de andere Limburgse teams het prima. Het Sint-Maartenscollege uit Maastricht werd vijfde, het Bisschoppelijk College Weert zesde en Jeanne d'Arc, ook uit Maastricht, zevende. Daarna pas kwamen de eerste Vlaamse deelnemers. Zij hebben een handicap, omdat ze minder toepassingsgerichte wiskunde krijgen. Dat bleek bijvoorbeeld bij de vraag hoe je een vel karton van 20 bij 32 centimeter zo moet vouwen en snijden dat je pakken krijgt waar zoveel mogelijk hagelslag in gaat.

De Limburger, January 27, 1997

2.8 De Limburger: a report on the 3rd MMM

The third edition of the MMM-competition, January 25, 1997, was reported in the Dutch regional newspaper *De Limburger*. The newspaper thought it necessary to observe that the first Flemish team was found on position 7 in the final ranking. *De Limburger* also suggested a possible explanation, namely that the mathematics taught at Flemish schools was less application oriented. In particular, Flemish students may be unfamiliar with 'hagelslag'.

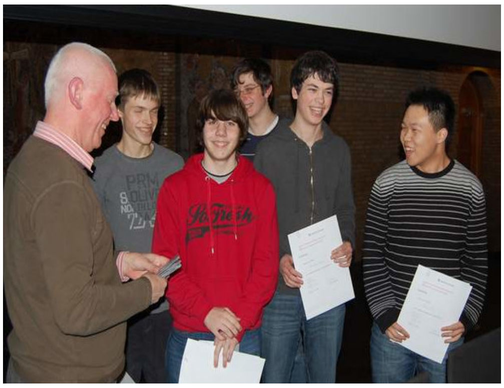
2.9 The prize winning schools

The table below shows the prize winning schools in the previous 24 editions of the MMM competition.

E.	1st prize	2d prize	3rd prize
1	Lorentz Lyceum Eindhoven		
2	Katholiek Gelders L Arnhem	Lorentz Lyceum Eindhoven	
3	Katholiek Gelders L Arnhem	St Janscollege Hoensbroek	
4	St Augustinusinstituut Bree	Bissch C Broekhin Roermond	St Jozef C Turnh *and* Mont C Mtricht
5	Montessori C Maastricht	St Aloysiuscollege Geel	Sted H Dilsen *and* Theresia L Tilburg
6	Stedelijke Humaniora Dilsen	KAI Oostende	Maaswaal College Wychen
7	Koninklijk Atheneum I Hasselt	Pontes Scholengroep Goes	Montessori College Maastricht
8	Elzendaal College Boxmeer	Freiherr von Stein G Leverkusen	Suitbertus G Düsseldorf
9	G Camphusianum Gorinchem	Stiftisches G Düren *and* Stedelijk L Antwerpen	
10	Het Goese Lyseum Goes	G Camphusianum Gorinchem	Heilig Hart Instituut Heverlee
11	Erzb Suitbertus G Düsseldorf	G Fabritianum Krefeld	Bonnefantencollege Maastricht
12	KOGEKA 6 Geel	Heilig Drievuldigheidsc Leuven	Suitbertus-G Düsseldorf
13	Aloisiuskolleg Bonn-Bad Godesberg	Humaniora Kindsheid Jesu Hasselt	WICO campus St-Hubertus Neerpelt
14	Sted Humaniora Dilsen-Stokkem	Wilhelm-Dörpfeld-G Wuppertal	Marienschule Leverkusen
15	Aloisiuskolleg Bonn-Bad Godesberg	C Heilig Kruis St-Ursula 2 Maaseik	Ernst-Moritz-Arndt-G Bonn
16	Ernst-Moritz-Arndt-G Bonn	Aloisiuskolleg Bonn-Bad Godesberg	Marienschule Leverkusen
17	Ernst-Moritz-Arndt-G Bonn	Aloisiuskolleg Bonn-Bad Godesberg	WICO Campus St-Hubertus Neerpelt
18	Ernst-Moritz-Arndt-G Bonn	Gymnasium Borbeck Essen	WICO Campus St-Hubertus Neerpelt
19	Silverberg G Bedburg	Städtisches Stiftsgymnasium Xanten	Vinzenz-Palotti-Koleg Rheinbach
20	Sint-Michielscollege Schoten	Ernst-Moritz-Arndt-G Bonn	Freiherr-vom-Stein-G Leverkusen
21	Sint Pieterscollege Leuven	Freiherr-vom-Stein-G Leverkusen	Sint-Michielscollege Schoten
22	Het College Weert	Sint Pieterscollege Leuven	Humaniora Kindsheid Jesu Hasselt
23	Sint Pieterscollege Leuven	Humaniora Kindsheid Jesu Hasselt	Sint-Dimphnacollege Geel
24	Abteigymnasium Brauweiler	Wilhelm-Dörpfeld-G Wuppertal	Gymnasium Mariënberg Neuss

MMM prize winning schools (abbr.: G Gymnasium, C College)

One school, Ernst-Moritz-Arndt-Gymnasium from Bonn, managed to get the first prize in three consecutive years, 2010 to 2012.



2011

2012

Winning teams from the Ernst-Moritz-Arndt-Gymnasium, Bonn

2.10 Are the MMM exercises difficult?

Each edition of the MMM competition contains 5 exercises, and with each of them 10 points can be gained. The table below shows how many points were needed to obtain a prize in each edition (no results were stored for the first four).

E.	year	points		
5	1999	47	46	44
6	2000	42	35	33
7	2001	32	31	30
8	2002	40	38	35
9	2003	32	30	30
10	2004	37	36	33
11	2005	42	38	37
12	2006	47	45	44
13	2007	39	38	32
14	2008	43	42	41
15	2009	44	34	32
16	2010	44	34	26
17	2011	36	33	32
18	2012	48	42	41
19	2013	50	49	45
20	2014	40	39	36
21	2015	46	45	44
22	2016	50	45	38
23	2017	43	31	27
24	2018	37	33	32

Points of the winning teams

Winning team in 2016, Het College Weert

Winning team in 2013, from Silverberg Gymnasium, Bedburg

Two times, in 2013 and 2016, the winning team managed to solve all exercises perfectly!

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1 euro = 100 cents = 10 cents x 10 cents = 1/10 euro x 1/10 euro = 1/100 euro = 1 cent. What went wrong?

3. Old questions, new ideas

Looking at the mathematical problems that have featured in the first 25 editions of the MMM, there are many which catch the eye. Some are funny for the context in which they are presented, others stand out because of their element of surprise. At first glance, some will seem quite impossible to solve; yet with a bit of patience, a way to approach the problem may suddenly pop up. Often, but not always, one may find a quick and elegant solution by merely looking at the problem from a less conventional point of view. All these problems, riddles, and puzzles of the MMM belong to the domain of *recreational mathematics*. The topics they touch upon are highly diverse, as most subfields of pure and applied mathematics lend themselves quite well for it, and also much inspiration can be drawn from physics, engineering, econometrics, and computer science.

Remove the nut from the bolt with the ball

The background of all this, is that in the applied sciences you often encounter situations in which you must make a decision. Either to achieve a certain goal, or to optimize something, or maybe you want a smart design, or you intend to develop a clever strategy. You can then, of course, base your choices on your intuition and experience alone and simply do whatever *seems* to be reasonable: you then take what is called a *heuristic* approach. But you can also try to model the situation mathematically, analyze it more carefully, and then go for an *optimal* approach. Depending on the complexity, this second approach often pays off and along the way, as an extra reward, you learn something and it can lead to interesting conclusions and deeper insight.

At the surface, many of the problems in recreational mathematics and of the MMM may seem to be just funny riddles (designed for those of us who appreciate that special kind of humor that comes along with them), but on closer inspection you will find that this is only part of their quality. They are often linked to much deeper theories and fundamental concepts in (pure) mathematics, and to symmetry and optimality principles in physics and in nature. Our intuition can all too easily be deceived, as every magician and mathematician knows. The purpose of this chapter then, is to highlight a small selection of MMM problems from the past, to share some of their fun, to make some of these hidden connections and underlying principles explicit, and to help you generate new ideas. Enjoy!

3.1 Secretly in love

From the 6th MMM, 2000. Adrian has a crush on Barbara and would like to bring a love-letter to her home. He knows in which street she lives, but so far he hasn't been able to figure out at which number. All he knows is that the houses in her street have numbers which run from 8 up to 100. They then have the following conversation. Adrian asks "Is your house number larger than 50?"; Barbara answers but she lies. Next, Adrian asks: "Is your house number a multiple of 4?"; and Barbara lies again. Then Adrian asks: "Is your house number a square number?" This time Barbara speaks the truth. Finally, Adrian asks: "Does your house number start with the digit 3?" Barbara smilingly denies this, but this time we do not know whether or not she speaks the truth. Adrian then reacts by saying: "Now I know where you live!" But when he tells her the number, he is all wrong.

A street in Klaipeda, Lithuania

Question: at which number does Barbara live?

Now, to get you started, this is a relatively easy problem to solve, of a type which features quite regularly at the MMM. It's just a matter of straightforward bookkeeping, so if you bother taking up a pen and paper you will find the answer quickly. (If you want, you can check your steps on our website – the result is 81.) But even though it's simple, this problem is included here because it shows two essential qualities of recreational math problems in their purest form: (1) If you read the story a little too fast or superficial, your intuition will tell you that this problem is unsolvable, because of all the lying and uncertainty in the answers. This is the element of deception. The problem looks hard while in fact it is easy. You may then be tempted to choose an unnecessarily complicated approach to solve it. (2) All you need for your solution is elementary logic reasoning, no guessing and gambling required. In this case, just follow the logic of Adrian and then combine it with the information provided on the truth or falsehood of Barbara's answers. Uncovering this 'flow of information' is often essential in solving this kind of recreational math puzzles.

To solve the following problem, exactly the same principles of logic reasoning can be applied. But this one is quite a bit harder, because you need to put much more effort into accurately writing down the information flow (i.e., building a mathematical model of it).

From the 17th MMM, 2011. Each of three people is wearing a hat on which a natural number $(1, 2, 3, \ldots)$ is printed. All three can see the numbers on the other two hats, but not their own number. All are told that one of the numbers is equal to the sum of the other two. The following statements are made in the hearing of all, and consecutively.
A: I cannot deduce what my number is.
B: I cannot deduce what my number is.
C: I cannot deduce what my number is.
A: I can deduce that my number is 50.

Question: What are the numbers on the other two hats? Compute all possibilities.

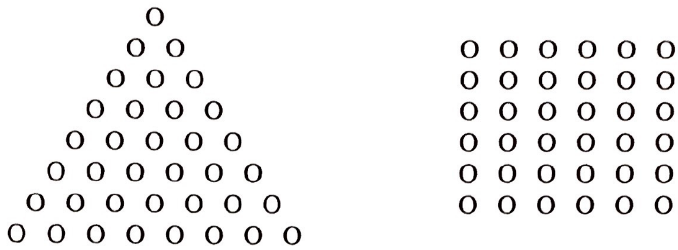
To solve this problem, consider the first statement of person A. He sees the numbers on the hats of B and C, but cannot deduce his own number. So we ask ourselves the question: when could he have deduced it? If we denote the numbers on the hats of A, B and C by the symbols a, b and c, respectively, then we know (and A knows) that either $a = b + c$, or a equals the (positive) difference between b and c, which we can write as $a = |b - c|$. The first option cannot be excluded. However, the second option can, *but only in the special case when* $b = c$, because then $|b - c| = 0$ while we know that $a > 0$ (as it is known to be a natural number). But A wasn't able to deduce the value of a, so we can conclude that $b \neq c$. And of course, B and C, who hear all this and are equally clever as A, can now make use of this extra information! This is the mathematical modeling that needed to be done here: the symbols a, b and c had to be introduced and statement A was rewritten, simply and efficiently, as the equivalent statement $b \neq c$.

In a similar way, we can proceed with the other statements, even though the complexity increases step by step. It follows that the first three statements give the following bits of extra information:
A: $b \neq c$.
B: $a \neq c$ and $c \neq \frac{1}{2}a$.
C: $a \neq b$ and $a \neq \frac{1}{2}b$ and $b \neq \frac{1}{2}a$ and $b \neq \frac{3}{2}a$.

With all this information combined, there are now several situations in which A can deduce the value of a to support the fourth statement. He still cannot exclude the option $a = b + c$, but he can rule out the option $a = |b - c|$ if for the values of b and c (which he sees!) it follows that the corresponding a is to violate one of the pieces of information collected above. But we also know that $a = 50$, and that b and c are both positive integers. This simplifies our computations and we find that there is only one possibility left: $a = 50$, $b = 20$ and $c = 30$. (The condition which then gets violated is $a \neq \frac{1}{2}b$ should you suppose that $a = |b - c|$. So, for these values one can indeed conclude that $a = b + c = 50$.)

3.2 Triangle or square?

From the 8th MMM, 2002. Strolling along the beach, Robin has collected 36 beautiful seashells. She suddenly notices that she can arrange them in two different but both highly symmetric geometric patterns: (1) as an equilateral triangle (with 8 shells on each side), and (2) as a square (with 6 shells on each side). This happens because 36 can be written on the one hand as $n(n+1)/2$ and on the other hand as m^2, for two integer numbers n and m (here we take $n = 8$ and $m = 6$). Robin asks herself if there are any other pairs of numbers with the same property. This is indeed the case, in fact there are infinitely many such pairs.



36 = 1+2+3+4+5+6+7+8   36 = 6x6

Question (a): Find other integer numbers $n > 8$ and $m > 6$ for which it also holds that $n(n+1)/2 = m^2$.
Question (b): Find a formula which allows you to compute a new larger pair (n,m) with the given property from an available smaller one. You are allowed to present a recursive formula, but *not* a brute force computer program which systematically checks new options until one happens to fit the required specifications.

The equation $n(n+1)/2 = m^2$ for integers n and m is called a Diophantine equation. It is named after Diophantus of Alexandria, a famous mathematician who lived in the Hellenistic city of Alexandria, in Egypt, in the 3rd century AD. A system of Diophantine equations has less equations than unknowns, all with integer coefficients, but with the characterizing condition that *solutions for all the unknowns must be integer-valued*. Without this integrality condition, a curve (or surface, or hypersurface) of infinitely many real solutions will usually exist; but with the integrality condition imposed this will be far less clear. And: if some solutions happen to be known, it still may not be possible to find more, or to find them all. A Diophantine equation can be viewed geometrically: one then is looking for grid points on a given curve (or surface) described by the equation. Diophantine equations have been studied since antiquity, because integers appear in many practical problems and the *rational* numbers (the quotients of integers) have long been the preferred class of numbers to work with. Certain *real* numbers, such as square roots, were known to be irrational in India (in the 8th century BC) and in Greece (in the 5th century BC), but a proper mathematical construction of the set of real numbers is a 19th century invention. Diophantine equations have the reputation that many of them are notoriously hard to solve. (But hey, this is the MMM, so we will not easily be scared.)

Here is a short list of some famous Diophantine equations:
$a^2 + b^2 = c^2$, describing all the *Pythagorean triples*. This produces all the right triangles having only sides of integer length. All solutions to this equation can be found by a similar approach as what we will use below to answer Robin's questions.
$a^n + b^n = c^n$, in which $n \geq 3$ is also an integer. *Fermat's Last Theorem* (1637) states that

this equation has no nontrivial integer solutions. After more than 350 years, this claim was finally proved to be valid, by Andrew Wiles in 1994.
$ax + by = c$, for given integer coefficients a, b and c, and integer unknowns x and y. This is a linear Diophantine equation, which corresponds to the *Bézout identity*. It states that solutions exist if and only if c is a multiple of the greatest common divisor of a and b. The *Chinese remainder theorem* generalizes this to more coefficients and more unknowns. Linear Diophantine equations have been completely solved.
$x^2 - ny^2 = \pm 1$, in which n is a given integer and x and y are the integer unknowns. Today this is called *Pell's equation*, after John Pell who lived in the 17th century. But it was already studied long before it got this name, notably by the mathematician Brahmagupta in the 7th century AD in India. The special case with $n = 2$, which we will encounter shortly, was in fact studied in India and Greece as early as the year 400 BC, because of its connection to approximations of $\sqrt{2}$ by rational numbers.

Brahmagupta

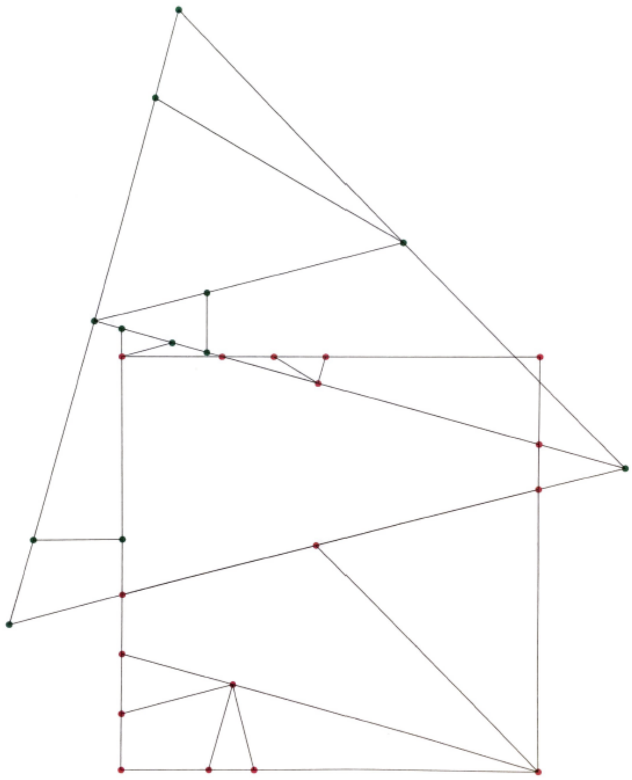
Let us return to Robin's problem and see if we can find a solution to question (b). A surprising feature of the given exercise is that we can do so with very elementary tools. We rewrite $n(n+1)/2 = m^2$ as $n(n+1) = 2m^2$, and we note that n and $n+1$ differ by just 1, so that they cannot have any factors in common. The prime factors of $2m^2$ come in even amounts, except for factors 2 of which there is an odd number. It follows that there exist integers k and ℓ (with no factors in common) such that $m = k\ell$ and either $n = 2k^2$ and $n+1 = \ell^2$, or $n = k^2$ and $n+1 = 2\ell^2$. This means that either $\ell^2 - 2k^2 = 1$, or $k^2 - 2\ell^2 = -1$ (so here we encounter Pell's equation for $n = 2$).

Now consider just the first case. We rewrite $\ell^2 - 2k^2 = 1$ as $\ell^2 - 1 = 2k^2$, then as $(\ell - 1)(\ell + 1) = 2k^2$. Because the right-hand side is even, ℓ is odd, which can be expressed by writing $\ell = 2p + 1$ for some integer p. We find that $2p(2p+2) = 2k^2$. The left-hand side has at least two factors 2, so that k is even. This can be expressed by writing $k = 2q$ for some integer q. Plugging this into the equation and dividing both sides by 4 produces: $p(p+1) = 2q^2$.

Here is the surprising element: starting from $n(n+1) = 2m^2$ we have arrived at $p(p+1) = 2q^2$, which appears to be *exactly the same problem*! Have we made any progress then at all? Yes we have, because p and q are *smaller* than the numbers n and m that we started from. If we select a given solution for p and q (for instance $p = 8$ and $q = 6$) then we can *retrace our steps* to compute a new and much larger solution for n and m. In this way we find $k = 2q = 12$, $\ell = 2p + 1 = 17$, for which $n = 2k^2 = 288$ and $m = k\ell = 204$. More generally: if (p,q) is a solution, then (n,m) is also a solution for $n = 8q^2$ and $m = 2q(2p+1)$.
Repeating this procedure, infinitely many solutions are obtained. (In this way, we don't find all solutions because we disregarded one case, but this is good enough to answer

Robin's questions. See the solutions on the MMM-website for more details.)

Bonus problem. The most natural context to pose the question of how to rearrange an equilateral triangle as a square, is of course not number theory, as in Robin's problem, but geometry. So, if you are given a square with sides of unit length, can you split it up, using only a ruler-and-compass construction, into pieces that can be rearranged to form an equilateral triangle? And what if you start from an equilateral triangle? Below (next page) is a figure that shows a solution, but the actual construction is left for yourself to discover.



Ruler-and-compass construction

3.3 How to catch a mouse?

From the 13th MMM, 2007. It is Thursday morning, and you have seen a mouse on your attic. There are two mouse holes in one of the walls: one on the left and one on the right. You would like to catch the mouse, and preferably before Sunday, because then your grandmother will visit you and mice scare her to death. The problem, however, is that you only have a single mouse trap. To catch the mouse you place the trap in front of one of the two holes, and then you leave it there for a whole day. The next day, if you have not yet caught the mouse, you may leave the trap where it is, or you may move it over to the other hole. Let us assume that during the day the mouse is lazy and does not move from one hole to the other.

If you placed the trap in front of the hole where the mouse is residing, then sometime during the day it will peep out and you will catch it with certainty. If it's placed at the wrong hole, you won't catch it. However, during the night the mouse doesn't come out, but it may walk from the vicinity of one hole to the other through some hidden corridors behind the wall.

You hold the following theory about the mouse: At Thursday morning, you believe there is a chance of 70% that the mouse is in the left hole, and 30% that it is in the right hole. If the mouse, on any day, is in the left hole and survives till the evening, you believe that there is a chance of 80% that it will go to the right hole during the night. If the mouse, on any day, is in the right hole and survives till the evening, you believe there is a chance of 50% that it will walk to the left hole during the night.

Question: Make a plan such that your chances of catching the mouse before Sunday are as big as possible.

This is an example of a strategy game with one player (that's you, not the mouse!) in which you also have to deal with uncertainty. Humans have a reputation for being really bad at dealing with uncertainty, particularly when based on intuition alone. We are all too easily scared by those rare incidents which make it to the headlines in the news (plane crashes, terrorist attacks, and so on) while we happily engage in far more dangerous activities on a daily basis (driving and biking on the road, smoking and drinking, playing sports). Did you know that most accidents occur at home, most murders are committed by relatives and friends, and that the mammal causing the highest amount of casualties each year is the cow?

So, what does your intuition tell you about this problem? Initially, the mouse is likely to be in the left hole, and you need to catch it fast. So if you were given just a single day to accomplish this task, then you should place the mouse trap on the left. Three days isn't that long, and you can still fiddle around with where you place the mouse trap on the other two days to further boost your chances. This is quite obvious how your strategy should look, you will think, so we must start on the left. Or must we?

A strategy like this is called a *greedy* heuristic. It's not optimal, but certainly not plain stupid either, because it is optimal for the first day. We just avoided any deeper analysis and we did not account for more complicated developments that could occur on day two and three. This is convenient, fast, and often gives reasonably good results, so we use it all the time in real life. In science and engineering we also use greedy strategies, for instance when we have to make quick decisions and a situation is too complex to fully analyze in a short time. Quite often, a 'smart technology' in artificial intelligence is really little more than a good and greedy heuristic, based on data analysis and pattern finding. But it's also a bit like the government not willing to plan several years ahead, because the next elections are coming up soon, and much can happen along the way...

To find an optimal strategy to catch the mouse, here we can simply perform a full analysis without too much effort. There are only 8 strategies to consider: on Thursday, Friday and Saturday, we can place the mouse trap either on the left or on the right: LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR. Each of these cases is easily analyzed taking the given probabilities into account. And even though LRR gives a catching probability of 97%, the optimal strategy surprisingly turns out to be RRR, which achieves 97.2%.

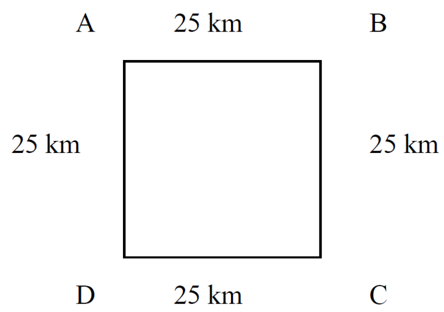
3.4 On the road again

From the 6th MMM, 2000. Four cities are located on the corners of a square with each side of length 25km. To economically strengthen the local region, a new network of highways is to be developed to replace the old one. It must allow the inhabitants to travel between all four cities avoiding smaller local roads.

For reasons of sustainability, the environment, and the economic use of resources, the total length of the new highways making up this new network has to be minimized.

Question: design such a minimal network and compute its total length.

A first way to go about this, a very simple one, is to connect A to B, B to C, and C to D, along the edges of the square they form. That makes 75km in total.



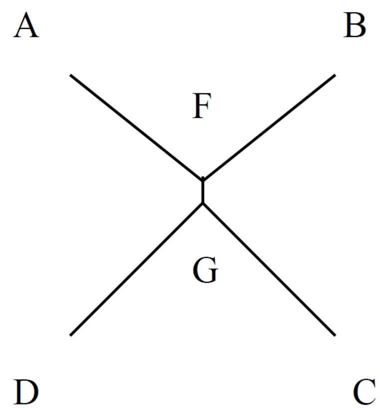
A networks of highways, 75 km in total

A second way, is to label the intersection point of the two diagonals E, and to connect A to E, B to E, C to E, and D to E. Then the total length of the network equals $50\sqrt{2} \approx 70.71$km, which is a few km less. But: it is still sub-optimal (which is a nice way of saying that you can do better)!

As it happens often in mathematics, you can gather useful insight by *studying simpler versions of the problem first*. In the present case, this means that you ask yourself the question what happens when you try to connect fewer cities, for instance two (well, that's easy!) or three.

For three cities, you then stumble into a famous problem raised by Pierre de Fermat in 1643 and solved by Evangelista Torricelli. The Fermat-Torricelli point of a triangle (with angles all less than 120 degrees), is the unique point inside that triangle such that the line segments which connect it to the three vertices make three angles of 120 degrees each. You can find it experimentally by exploiting a principle from physics which states that nature strives to minimize potential energy. Draw the triangle on a piece of cardboard, hold it flat, drill small holes at the locations of the vertices, put a separate piece of string through each hole and tie each string to a separate but equal weight below the cardboard, finally tie the ends of the three strings above the cardboard together in a single knot, and then let go. As nature minimizes the potential energy in the system, the total length of the three pieces of string below the cardboard is maximized, so the total length of string above the cardboard gets minimized. The knot above the cardboard will therefore move to the location of the Fermat-Torricelli point. Since the three weights are equal, the forces in the three strings are equal too, so at the knot there are three equal forces pulling at it into three different directions. For the knot to be in equilibrium, these forces must pull into directions which make angles of 120 degrees. (Of course, you can also find this point by a suitable geometric construction, using ruler and compass; see the MMM-website.)

To see that you can improve on the second approach above, just consider the triangle formed by the vertices A and B and the intersection point of the diagonals, E. This part of the road network design can now be improved by connecting A, B, and E using the Fermat-Torricelli point (say F) of the triangle ABE. Likewise, the other part of the network involving C, D, and E can also be improved using the Fermat-Torricelli point (say G) of the triangle CDE. We then arrive at what turns out to be an optimal solution, for which the total length of the network is easily computed to equal $25(1+\sqrt{3}) \approx 68.30$km.



The optimal solution, about 68.30 km in total

This problem is an instance of what is known as the *Euclidean Steiner tree problem*. In general, when N locations are to be connected, one may have to introduce up to $N-2$ additional points (Steiner points, i.e., local Fermat-Torricelli points) to arrive at an optimal design which minimizes the total length of the network. In the given problem we had 4 points (A, B, C, and D) and we introduced 2 Steiner points (F and G; note that E is located on the line connecting F and G and can be omitted from the final solution). The Steiner tree problem has several generalizations to different situations of practical relevance. For instance, the Steiner tree problem *on weighted graphs* aims at connecting a choice of given nodes in a graph using only the edges (and possibly some more nodes) of that graph, thereby minimizing the total weight along the edges that are used. In this sense, it generalizes and combines two famous problems from combinatorial optimization and graph theory: the shortest path problem and the minimum spanning tree problem. The Euclidean Steiner tree problem can be generalized to points in n dimensions, and also to situations in which the paths are required to consist of sections that must run either horizontal or vertical, which is useful in micro-electronics for optimizing the design of computer chips.

3.5 A piece of cake

From the 14th MMM, 2008. A cake is sliced into a number of completely equal pieces. The cake is distributed among some cake-lovers by the following procedure, carried out by the cake-master. Each cake-lover is given a number of colored pins. Then each cake-lover distributes her pins (each person has a different color) over the slices in anyway she likes. If all the pins are stuck into the cake, and no cake-lover wishes to change the distribution of her pins, then each of the slices is distributed proportionally with respect to the numbers of pins stuck into it.
(For instance, if a particular slice has 2 red pins in it and 3 blue pins, then the owner of the red pins gets 2/5 of this slice and the owner of the blue pins gets 3/5.)

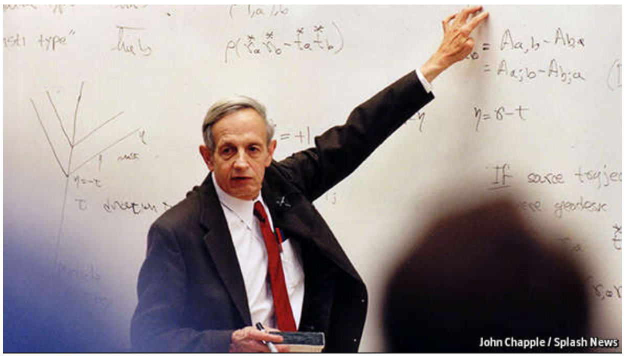
Carrot cake slices

Suppose that the cake is sliced into 10 pieces and that there are three cake-lovers, Red, Blue and Green: Red has 4 pins, Blue has 6 pins, and Green has 8 pins. The cake-lovers start distributing their pins over the slices. To her surprise, the cake-master observes that after a little while none of the cake-lovers wishes to change her pins.

Question (a): Which share of the cake does each cake-lover get? [Hint: more than one distribution is possible.]
Question (b): Suppose now that the cake is sliced into 3 equal pieces, and there are 6 cake-lovers with five of them having 2 pins and one of them having 1 pin. Will the cake-lovers ever stop changing their pins? If so, describe the possible shares of the cake that each one gets. If not, provide an argument why.

In this problem, we see different cake-lovers competing with each other to get as big a piece of the cake as they can. Apparently they can make initial choices, but afterwards they can still change their mind and respond to the choices the others are making. In the situation of question (a), at some moment in time they happen to stop making changes. What does this mean?
It means that the players (the cake-lovers) have found what is called a *Nash equilibrium* for the game they are playing (the cake-selection competition they are having). A Nash equilibrium is named after John Nash, an American mathematician (1928-2015), who is most famous for his ground-breaking work in game theory and decision-making. He shared the Nobel prize in economic sciences in 1994, and he shared the Abel prize in 2015 for his work on nonlinear partial differential equations. The 1998 biography and the 2001 movie *A Beautiful Mind* present an intriguing portrait of his life, scientific achievements, his mental illness and the resulting personal struggles.

We say that the players are together 'in Nash equilibrium', if for each player it holds that she cannot obtain a better outcome by unilaterally changing her decisions if the decisions of the other players all remain unchanged. Nash has proved the existence of a Nash equilibrium for a large class of games to which the current game also belongs: the situation observed by the cake-master is precisely that of a Nash equilibrium that has occurred.



John Nash

How to compute a Nash equilibrium for the game at hand? A brute force approach could work like this. Each of the players Red, Blue, and Green, has a finite number of strategies to deploy: there are just a finite number of pieces of cake into which they can stick the finite numbers of pins they have. For each possible combination of the choices they can make, you then can check what happens if one player were to deviate unilaterally. Though this approach will work in theory, this is not an advisable practical approach to the problem, because unfortunately there are way too many combinations to try. Therefore, one needs to start reasoning to exclude the majority of combinations and to limit the computational work.

First observation: There are 10 slices and in total 18 pins. Clearly, there will be at least 1 pin in every slice, otherwise any player would do better (get more cake) by moving one of her pins from a slice with multiple pins to this slice without pins.

Second observation: There will be no slice with 3 (or more) pins in it. This can be seen as follows. There must be a slice with only 1 pin in it. Consider a slice with at least 3 pins. Clearly, there are no multiple pins in it from one and the same player, otherwise that player could remove one of her pins and stick it elsewhere where she has no pins yet, thus getting more cake. Therefore, all three players have one pin in the slice with the 3 pins, and two of them have no pin in the slice with 1 pin. Such a player can do better by moving her pin from the slice with 3 pins to the slice with the 1 pin: she loses 1/3 of a slice but gains 1/2 of a slice.
Hence, every slice has either 1 or 2 pins in it.

Third observation: The eight slices with 2 pins in it each have pins of different colors in it: otherwise a player with 2 pins in one and the same slice can easily do better by sticking one of her pins into a different slice claimed by at least one of the other players.

Any configuration satisfying the requirements established so far will be a configuration where no player can do better by changing her own pins. There are many such configurations but in terms of shares of the cake there are only six different ones, depending on who has pins in the two slices that have only 1 pin in it. Denoting these by the respective colors, we have the following shares for (R,B,G):
RR: (3/10,3/10,4/10)
BB: (2/10,4/10,4/10)
GG: (2/10,3/10,5/10)
RB: (5/20,7/20,8/20)
RG: (5/20,6/20,9/20)
BG: (4/20,7/20,9/20)
This shows all the valid answers that can be given to question (a).

To address question (b), a similar approach can be taken. There are now 3 slices and 11 pins.

First observation: There can be no slice with 5 (or more) pins in it. For in that case, there must be a slice with (at most) 3 pins in it. Then it must be the case that all pins in the 5-pin slice are from different players (otherwise there would be a player with both pins in this slice and that player would do better by moving one of her pins to the 3-pin slice). At least one of them does not have a pin in the 3-pin slice, and can do better by moving her pin there (loss 1/5, gain 1/4).
Hence, there must be two slices with 4 pins and one slice with 3 pins.
Second observation: One can argue again that no player has both pins in one and the same slice.
This leaves many different possible configurations, but they are all of two kinds:
(1) The 1-pin player has her pin in the 3-pin slice: then the 1-pin player gets 4/36, two of the other players obtain 7/36 each, and the remaining three players obtain 6/36.
(2) The 1-pin player has her pin in a 4-pin slice: then she gets 3/36, three of the other players get 7/36 each, and the two remaining players get 6/36 each.
(In each of these cases we do have a Nash equilibrium, and there is no reason why the cake-lovers wouldn't find it or continue to change their pins.)

In game theory, the strategies that the players can use may sometimes also be *mixed* strategies. In the problem above, the players had to place their pins somewhere, which forced them to use pure strategies. If instead the players play the same game repeatedly, but in each round they have to place their pins simultaneously (thus being able to anticipate the other players' actions, but unable to respond immediately to what the others are playing currently), then they could decide to choose their actions *randomly* from their own set of pure strategies. This is what is called a mixed strategy. Again, such games can exhibit mixed Nash equilibria, for which one then must evaluate statistically the *expected* pay-offs.

Fortunately, not all food sharing competitions are as complicated as the one above; some are really a piece of cake.

From the 15th MMM, 2009. Alice and Bob are sharing a round pizza. Bob starts by cutting the pizza into an even number of slices, and all slices are circular sectors of possibly different angles. Then Alice continues by taking a piece of her choice, and after that they both take turns taking a piece. However, they are only allowed to take one of the two pieces that is at either end of the remaining part of the pizza.
Clearly, Bob could cut the pizza into equal slices and in that way obtain exactly half of the pizza.

Question: can Bob guarantee to obtain strictly more than half of the pizza?

If you think that the answer is yes, then you should describe a strategy for Bob that makes him get more than half of the pizza, no matter what Alice does. This means that you have to describe how Bob cuts the pizza, and how he proceeds after that, depending on which slices Alice takes.
If you think that the answer is no, then you have to show that, no matter how Bob cuts the pizza, Alice always has a way of picking pieces so that Bob gets at most one half of the pizza in total.

Here is a quick solution. Alice alternately colors the slices of the pizza black and white (in her mind, of course). Then either the black slices or the white slices comprise at least half of the area of the pizza.

Let us assume it is the black slices. Now Alice can start by taking any of the black slices. Then Bob only has two white slices to choose from. No matter which one he takes, Alice will have the choice between a black and a white slice, and will take the black one. Again, Bob can only choose from two white slices. This way Alice will end up with all the black slices and Bob gets all the white slices, giving Alice at least half of the area of the pizza.

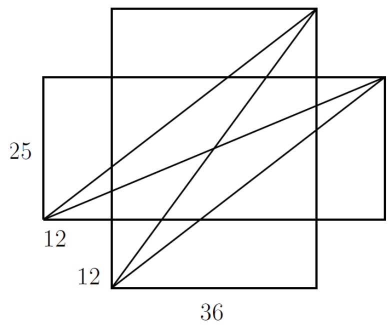
A funny element in this solution, is that Alice can start by taking the smallest slice of the color she wants to collect all the slices of, and still end up having at least half of the pizza! And of course, if Alice is on a diet and wants to downsize her pizza consumption, she can also guarantee to get at most half of the pizza.

3.6 Ant problems

It's that itchy feeling again. Ant problems! Did you know that, curiously, Ant-arctica is the only continent on earth without ants? Yes, ants are virtually everywhere, and the clever way in which they cooperate to solve complicated tasks are a source of inspiration for scientists in game theory as well as in artificial intelligence, to find and describe results of cooperative strategies for large groups of autonomously acting players or agents.

From the 20th MMM, 2014. A 12 by 25 by 36 cm cereal box is lying on the floor on one of its 25 by 36 cm faces. An ant, located at one of the bottom corners of the box, must crawl along the outside of the box to reach the opposite bottom corner.
(Note: The ant can walk on any of the five visible faces of the box, except for the bottom face, which is flath in contact with the floor. It can crawl along any of the edges. It cannot crawl under the box.)

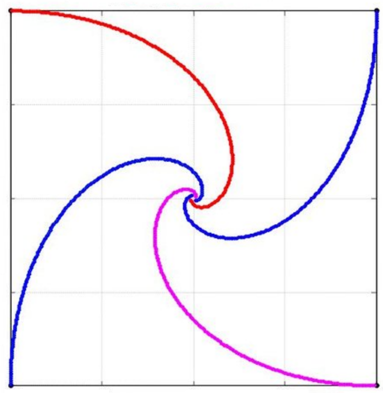
Question: What is the length of the shortest such path?



Now, this is not such a tough problem. To solve it, just flatten the box and lay it out, so that you can design a route by drawing a straight line from one point to another. There will be multiple points corresponding to the starting point and to the targeted end point, but you will see that there are just a few candidate routes to consider. The answer you are expected to find is $\sqrt{37^2+48^2} \approx 60.605$ cm.

Another classic ant problem. Today in my garden I witnessed 4 ants on a square tile of size 50 by 50 cm. Each of them was in a different corner when suddenly they saw each other and started to move. In a clockwise manner, the first ant walked straight into the direction of the second ant, the second towards the third, the third towards the fourth, and the fourth towards the first.
They noticed that their targets were moving, so they continuously adapted their directions to compensate for it, always keeping their focus straight on their targets. They all moved at the same speed of 2 cm/s, and eventually they all arrived together in the center of the tile.

How long is each of the paths the ants walked along (i.e., from the corner along the curved trajectory to the center of the tile)?



What makes this problem interesting, is that it is again about a shortest path, seems quite hard to model mathematically, but can be solved easily once you adopt the right point of view. What you could do, is to set up a differential equation which describes the curve a chosen ant is walking along, then solve that equation, and finally set up and solve an integral to compute the path length. That's hard and complicated and we will not do so here. Once again, physics is called to the rescue. Sometimes you can find shortest paths by considering how light will travel in varying media, as light is bound to take the (time-wise) shortest path. In this case, however, physics helps by exploiting the *symmetry* of the problem. For each of the 4 ants, the situation is identical, so at all times they constitute a perfectly square configuration. As they walk and adapt their directions, the ants make that square configuration smaller, and they also make it rotate. By adopting a coordinate system that counteracts the rotation, simply by rotating along with the ants (just imagine a suitably rotating camera above the scene, a trick often used in movies), what remains to be seen is a shrinking but otherwise stable square configuration. So how fast does it shrink?

From the perspective of an ant that is fully focused on its target, all it experiences is an initial distance of 50 cm and a speed of 2 cm/s in the chosen direction. The target ant moves into a direction that is perpendicular to the direction of the ant that hunts it, so from a relativistic perspective it does not delay nor speed up the hunting ant. Therefore: each ant needs 25 seconds to reach its target and they all walk curved trajectories that are 50 cm long.

As a final challenge: can you now figure out how long it takes if my garden tiles were triangular with sides of 50 cm and 3 ants in their corners? And what if the tiles were hexagonal with 6 ants on it?

Ralf Peeters, Department of Data Science and Knowledge Engineering, Maastricht University

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What is the mean distance between two random points on a unit square?

4. The Programme of the 25th MMM

The official announcement of the 25th MMM competition is found on the next page.

Audience at one of the lectures for teachers, 2011

It can be easily calculated that the digits 0 to 9 can be arranged into 3628800 distinct ten-digit numbers. How many of these numbers are prime?

25th *M*ATHEMATICAL *M*ODELLING COMPETITION *M*AASTRICHT

Date: Saturday January 26, 2019
Location: Tongersestraat 53, 6211 LM Maastricht
(You can park your car at the parking lot over there)
Room: Introduction in the Aula
(Building of the School of Business and Economics, Maastricht University)

PROGRAMME

10.00 - 11.15 hrs Reception and registration
NOTE: Sandwiches are offered again this year !

11.15 – 11.30 hrs Introduction (Aula)

11.30 - 14.00 hrs **Students:** Competition
Small bottles of water are available in the Mensa
to take away into your appointed room

Teachers: Lectures
Location: room H0.06, opposite of the Aula

11.30 hrs Lecture by: Niels Mourmans
Title: ‘Psychological games: modeling strategic

12.00 hrs Lecture by: Katerina Stankova
Title: ‘Cancer Therapy: Changing the game’
interactions with other-regarding motivations and
emotions’

12.30 hrs *COFFEE BREAK* (Ad Fundum)

12.45 hrs Lecture by: Gijs Schoenmakers
Title: ‘Big, bigger, biggest. Some of the largest
numbers ever used in mathematics!’

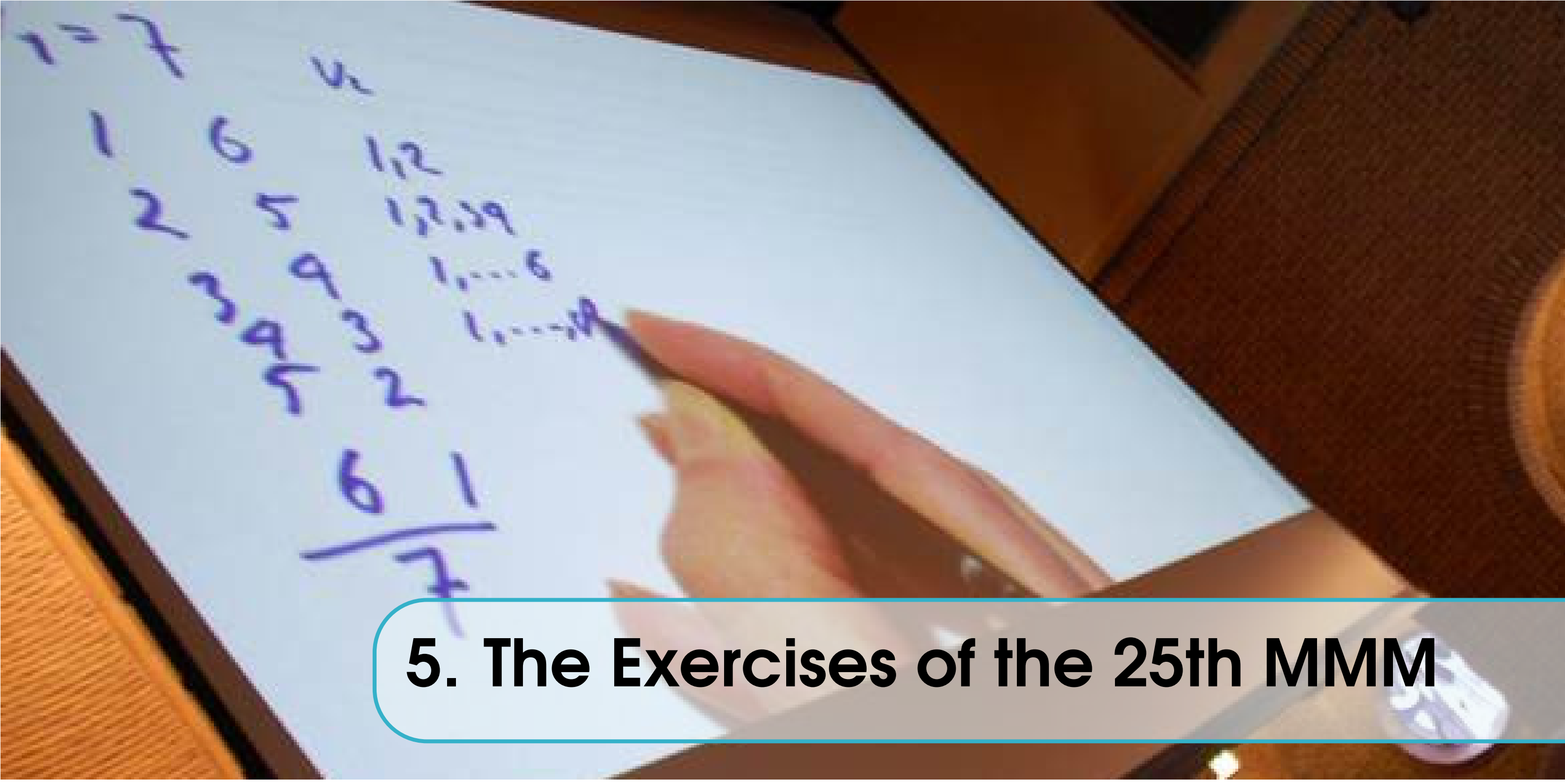
13.15 hrs Lecture by: Christof Defryn
Title: ‘The supermarket math’

13.45 hrs END

14.00 - 15.00 hrs Break (drinks) for everybody in the Mensa

15.00 - 16.00 hrs Explanation about the exercises and prize ceremony (Aula)

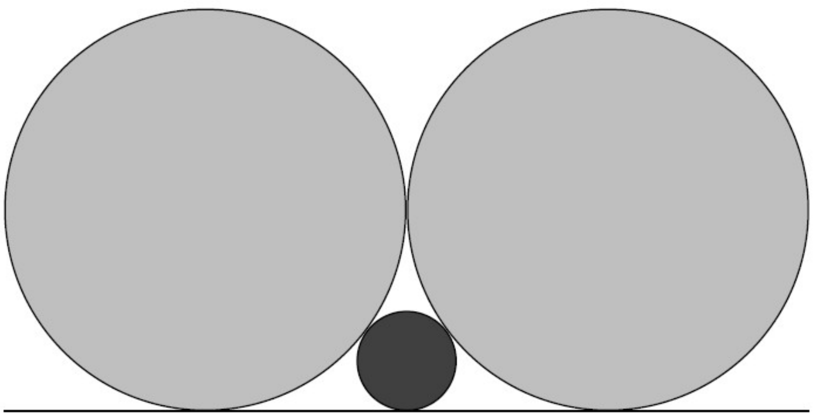
Information MMM: https://project.dke.maastrichtuniversity.nl/mmm/.



5. The Exercises of the 25th MMM

5.1 Problem 1: Three Circles

The three circles in the figure below touch each other pairwise and they all touch the bottom line. The two light gray discs each have an area of 25. What is the area of the dark gray disc?



5.2 Problem 2: Weight Watching

A huge bag contains infinitely many balls of different weights. For each non-negative integer $n \geq 0$, it contains 25 balls of weight n. Thus, there are 25 balls of weight 0, 25 balls of weight 1, 25 balls of weight 2, etc. Let us assume that days are numbered: $d = 0, 1, 2, 3, \ldots$

Is it possible to take out exactly five balls on each day d (until forever), such that the total weight of the balls taken out of the bag on day d is equal to d? For example, on day 1 the only way to do that is to pick four balls of weight 0 and one ball of weight 1.

If you think that this is possible, then show how. Otherwise, give an argument why this is not possible.

5.3 Problem 3: Line Dancing

Professor K makes a euphoric dance after making another scientific breakthrough. The dance consists of 25 equally sized steps, 14 to the left and 11 to the right. So, indeed professor K is performing a line dance. Obviously, at the end of his little dance Professor K is three steps to the left of his starting position.

a. How many different possibilities are there for professor K's dance?
b. How many of those dances start with a step to the right?
c. It turns out that, once he started his little dance, the professor never visited his starting position anymore. How many different possibilities are there now for professor K's dance?

5.4 Problem 4: Cards in Space

Lando bets his spaceship against Han in a party of the most famous card game in the universe. We shuffle 24 cards, numbered from 1 to 24, then we deal half of the cards to Lando and half of the cards to Han. One at a time, they play one of their cards on the table, with the number visible. The winner is the first player who plays a card such that the sum of the cards on the table is divisible by 25. If Lando starts, and if each player plays optimally, what is the probability that Han wins (and then becomes the new owner of the Falcon Millenium)?

5.5 Problem 5: 'Goes out one ear and into the other'

Today is party time, as we celebrate the 25th edition of the MMM. My garden gnome participates in these festivities, and I have decorated it with a special red party hat, which carries the five-pointed star of the city of Maastricht that I placed right on the front, with its center exactly in the middle of the party hat.

The upper part of the garden gnome's head (which for the most part fortunately is hidden under the hat) is shaped as a sphere, and the gnome's two ear holes happen to be located precisely diametrically opposite each other on that sphere. When measured in centimeters, the distance between the two ear holes (the diameter of the sphere) exactly equals the square root of 136.

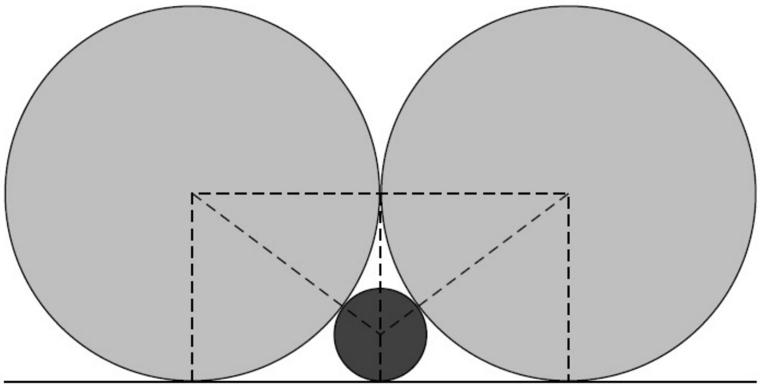
The party hat itself is shaped as a cone and it perfectly fits on the garden gnome's head. To specify this further: it is tangent to that sphere just there where the brim of the hat touches it in a circle. This circle is of course a little smaller than the 'great circles' on the sphere: if the garden gnome stands straight up, then this touching circle is placed in a horizontal plane exactly 2 cm above the horizontal plane that passes through the two ear holes.

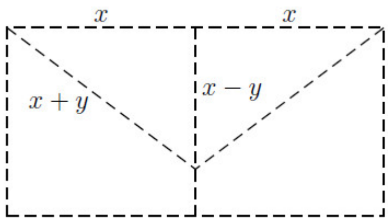
This morning I witnessed a surprising scene. An ant crawled out of the garden gnome's left ear! It first crawled up to the brim of the party hat to the point (2 cm higher) that is straight above the left ear. I later measured that in doing so, it covered a distance of 2.51 cm. (Clearly, it needed a bit more than 2 cm to get there, because as you can see the ears aren't nicely shaped as a sphere.) Next, from that point on the brim onward, the ant crawled further across the party hat in the shortest way possible to the center of the five-pointed star of Maastricht. Then it continued its way to the right ear, in which it finally disappeared, by following a path which precisely mirrored the path it took on the left side of the garden gnome's head and hat.

Question (a): My garden gnome is English, and it's indicated on the label that without the hat it's exactly 20.60 inches tall. How tall is it, in inches, now that it is wearing the hat?

Question (b): What is the total distance covered by the ant, in centimeters, when it went out one ear and into the other, along the path described above?

(Note: By definition, 1 inch exactly equals 2.54 cm. For both questions: specify your answers to 2 decimals after the decimal point.)





5.6 Solutions

Answer to Problem 1: Three Circles

Here, x is the radius of the big circle. Since $\pi x^2 = 25$, $x = 5/\sqrt{\pi}$.
The radius of the small circle is y. By the Theorem of Pythagoras, $(x+y)^2 = x^2 + (x-y)^2$, hence $x^2 = 4xy$, so that $y = x/4 = 5/(4\sqrt{\pi})$. Therefore, the area of the dark gray disc is $\pi y^2 = 25/16 = 1.5625$.

Answer to Problem 2: Weight Watching

This is possible, as follows.

$d = 0$: $0,0,0,0,0$
$d = 1$: $1,0,0,0,0$
$d = 2$: $2,0,0,0,0$
$d = 3$: $3,0,0,0,0$
$d = 4$: $4,0,0,0,0$

$d = 5$: $1,1,1,1,1$
$d = 6$: $2,1,1,1,1$
$d = 7$: $3,1,1,1,1$
$d = 8$: $4,1,1,1,1$
$d = 9$: $5,1,1,1,1$

and in general for $d = 5x + y$, with $x = 0,1,\ldots$ and $y = 0,1,2,3,4$:

$d = 5x + 0$: $x + 0,x,x,x,x$
$d = 5x + 1$: $x + 1,x,x,x,x$
$d = 5x + 2$: $x + 2,x,x,x,x$

$d = 5x+3$: $x+3, x, x, x, x$
$d = 5x+4$: $x+4, x, x, x, x$

This way we use 21 balls of weight 0, 22 balls of weight 1, 23 balls of weight 2, 24 balls of weight 3, and 25 balls of weight 4 or larger. To see this, note that we use balls of weight $x \geq 4$ on days $d = 5x+0, 5x+1, 5x+2, 5x+3, 5x+4$ (in total 21 balls), and on days $5x-4, 5x-8, 5x-12, 5x-16$ (in total 4).

Answer to Problem 3: Line Dancing

a. A dance consists of 25 steps of which 14 are to the left. So we select 14 from the 25 steps that are going to be the steps to the left. This can be done in $\binom{25}{14}$ $(= 4457400)$ ways.

b. After the step to the right we still have 24 steps to go, 14 of which should be to the left. So $\binom{24}{14}$ $(= 1961256)$ dances start with a step to the right.

c. Notice that we can write a dance as a sequence of L and R of length 25, consisting of 14 Ls and 11 Rs. We have three different types of dances: the ones that start with an R, the ones that start with an L and that do return to the initial position some time during the dance and the ones that don't return to the initial position during the dance.

We need the following observation. Suppose that we have a dance d that starts with an R. Then after the first step prof. K has taken more steps to the right that to the left. Since at the end of the dance he has taken more steps to the left, we know that after some number of steps, say $p \in \{2, 4, \ldots, 22\}$ we have that for the last time he has taken exactly the same number of steps to the left as to the right.

Now consider dance $\hat{d}$, that is the 'complement' of dance d in the sense that at all steps until step p it takes a step in the opposite direction of dance d and after step p the steps are in the same direction as dance d. An example (for an eight-step dance): If $d = RRLLLRLL$, then $p = 6$, so the first 6 steps of $\hat{d}$ must be the opposite of d. So $\hat{d} = LLRRRLLL$. We can do this for every dance d that starts with an R, which shows that the number of dances starting with an R is exactly the same as the number of dances starting with an L that does return to the initial position some time during the dance.

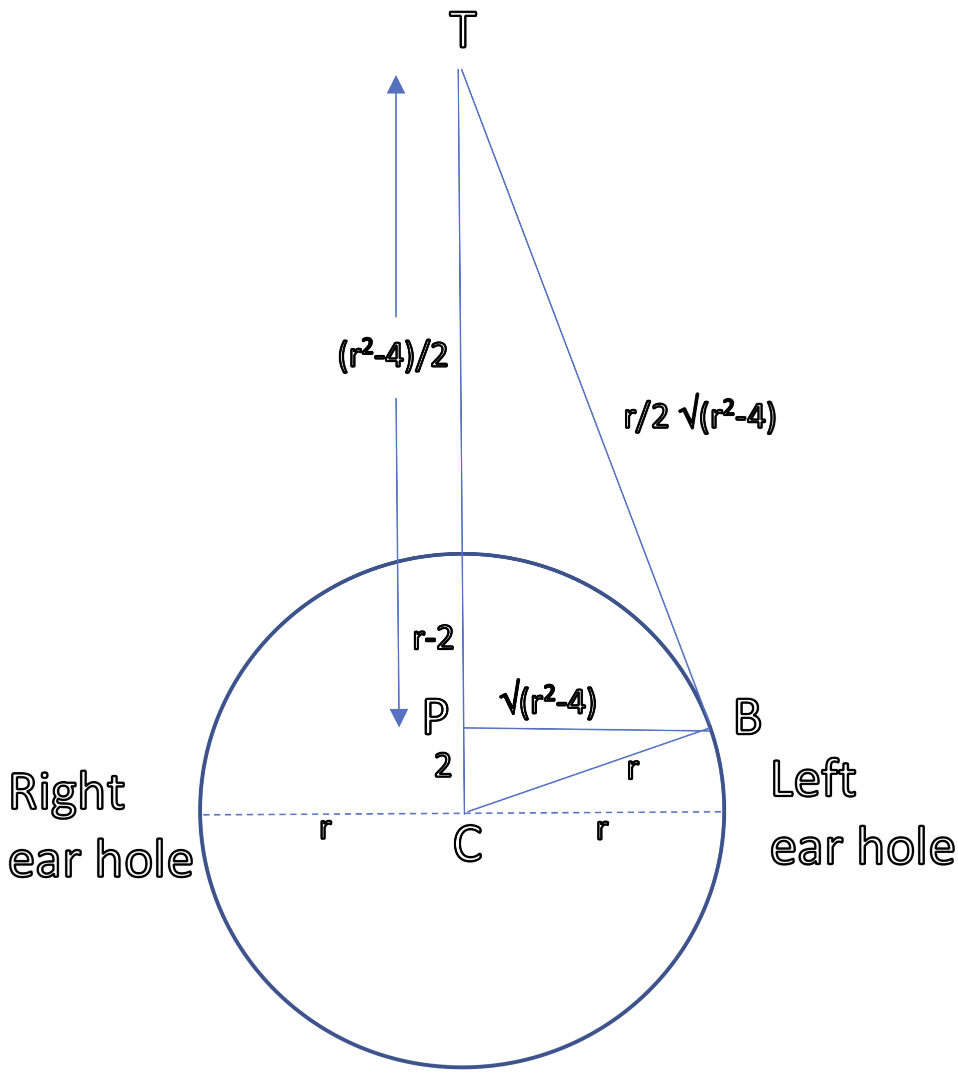
Therefore, let X be the number of dances that start with L and that do not return to its initial position during the dance. Then X is equal to the total number of dances minus two times the number of dances that start with an R. Or: $X = \binom{25}{14} - 2 \cdot \binom{24}{14} = 4457400 - 2 \cdot 1961256 = 534888$.

Answer to Problem 4: Cards in Space

Remark that Lando cannot win the first card, because the numbers between 1 and 24 are not divisible by 25. After the first card has been played, the remainder of the division by 25 of the sum of the cards on the table is a number i between 1 and 24. Only the card $24-i$ can win at the next choice of Han (but maybe Han does not have it). Then, for each card he has played, Han knows the card Lando needs to have to win at the next turn. Moreover, Han knows the cards of Lando (all the numbers between 1 and 24 except the cards he had at the beginning and the cards they played). As Lando started Han has one more card than Lando when he plays, so he can choose a card such that Lando cannot win the next turn. Hence, Lando cannot win. Then, either Han wins before he plays his last card, or he plays his last card, and the sum $1+2+\ldots+24 = 24*25/2$ is divisible by 25. So Han wins all the time, and the probability is 1.

Solution 5, 'Goes out one ear and into the other'

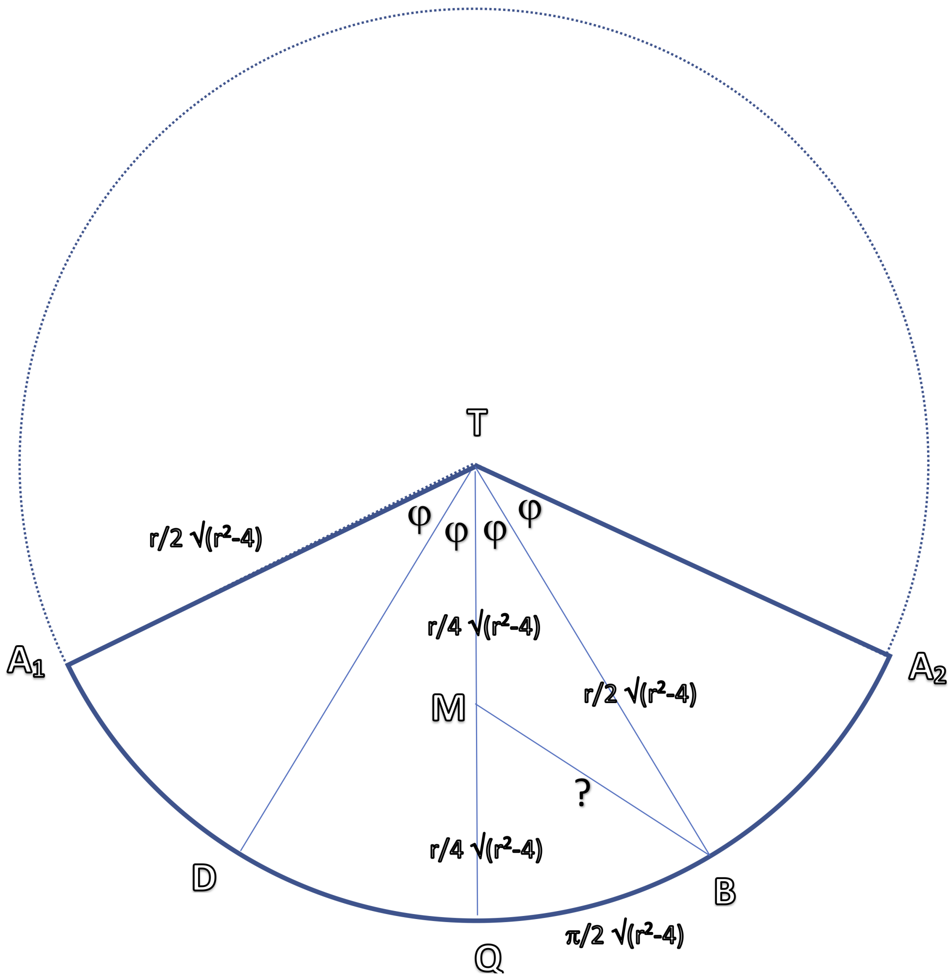
Question (a): When viewed from the front, the situation is as follows.



Working in centimeters, the diameter of the sphere equals $\sqrt{136}$, so that $r = \sqrt{34}$. From the center point C to the point B where the brim of the party hat touches the head of the garden gnome right above the left ear, the distance is also equal to r. Since B is in a horizontal plane that is 2 cm above the horizontal plane passing through the two ear holes, the center point P of the circle that constitutes the entire brim of the party hat, is located 2 cm above the point C. The angle BPC is a right angle, so according to Pythagoras' law the radius of this circular brim of the party hat equals $\sqrt{r^2-4} = \sqrt{30}$.

The angle TBC is a right angle, because the party hat fits precisely, so it is tangent to the spherical head at the point B. Therefore, we have that the three triangles TBC, BPC, and TPB are all of the same shape (congruent), which allows us to calculate the distance $|TB|$ as $(\sqrt{34}\sqrt{30})/2$ cm and the distance $|TP|$ as $(r^2-4)/2 = 15$ cm. Finally, it follows that the distance $|TC|$ equals $2+(r^2-4)/2 = 17$ cm, so that the top of the party hat is $17-\sqrt{34}$ cm above the top of the head of my garden gnome. Converting this to inches shows that my garden gnome with party hat is $(17-\sqrt{34})/2.54 = 4.40$ inches taller than 20.60 inches, so the answer is 25.00 inches.

Question(b): To determine the shortest path on the party hat, you should realize that a party hat is made every day by children in this world, cutting it from a flat piece of paper and bending it together. If you were to take a pair of scissors to cut it open at the back, the paper hat would simply flatten and take the shape displayed in the next figure. The sides TA_1 and TA_2 indicate the place where you used the scissors; the point



M indicates the center point of the five-pointed star of Maastricht (in the middle!) and the ant is located above the left ear, precisely at the point B on the brim of the hat. On this flat piece of paper you need to compute the length of BM. The points $A_1 = A_2$, B, Q, and D, are 4 points that divide the brim of the party hat into 4 equal pieces; Q is right in front. The total length of the brim follows from its radius before cutting the party hat open (see question (a)): it equals $2\pi\sqrt{r^2-4} = 2\pi\sqrt{30}$. The total length of the full circle with radius $\frac{r}{2}\sqrt{r^2-4}$ from which the flattened paper hat is just a sector, equals $\pi r\sqrt{r^2-4}$, which is $r/2$ times larger. So: 2π (the angle of a full circle) is $r/2$ times larger than 4φ (the angle of the sector) as the circumferences are proportional to the angles. This means that the top angle 4φ of the sector is $4\pi/r$ radians, which gives: $\varphi = \pi/r = \pi/\sqrt{34}$ radians.

To compute |BM| we now consider the triangle TBM and we use the "law of cosines" (which generalizes Pythagoras' law from right triangles to arbitrary triangles):

$$|\mathrm{BM}|^2 = |\mathrm{TM}|^2 + |\mathrm{TB}|^2 - 2|\mathrm{TM}| \cdot |\mathrm{TB}| cos(\varphi).$$

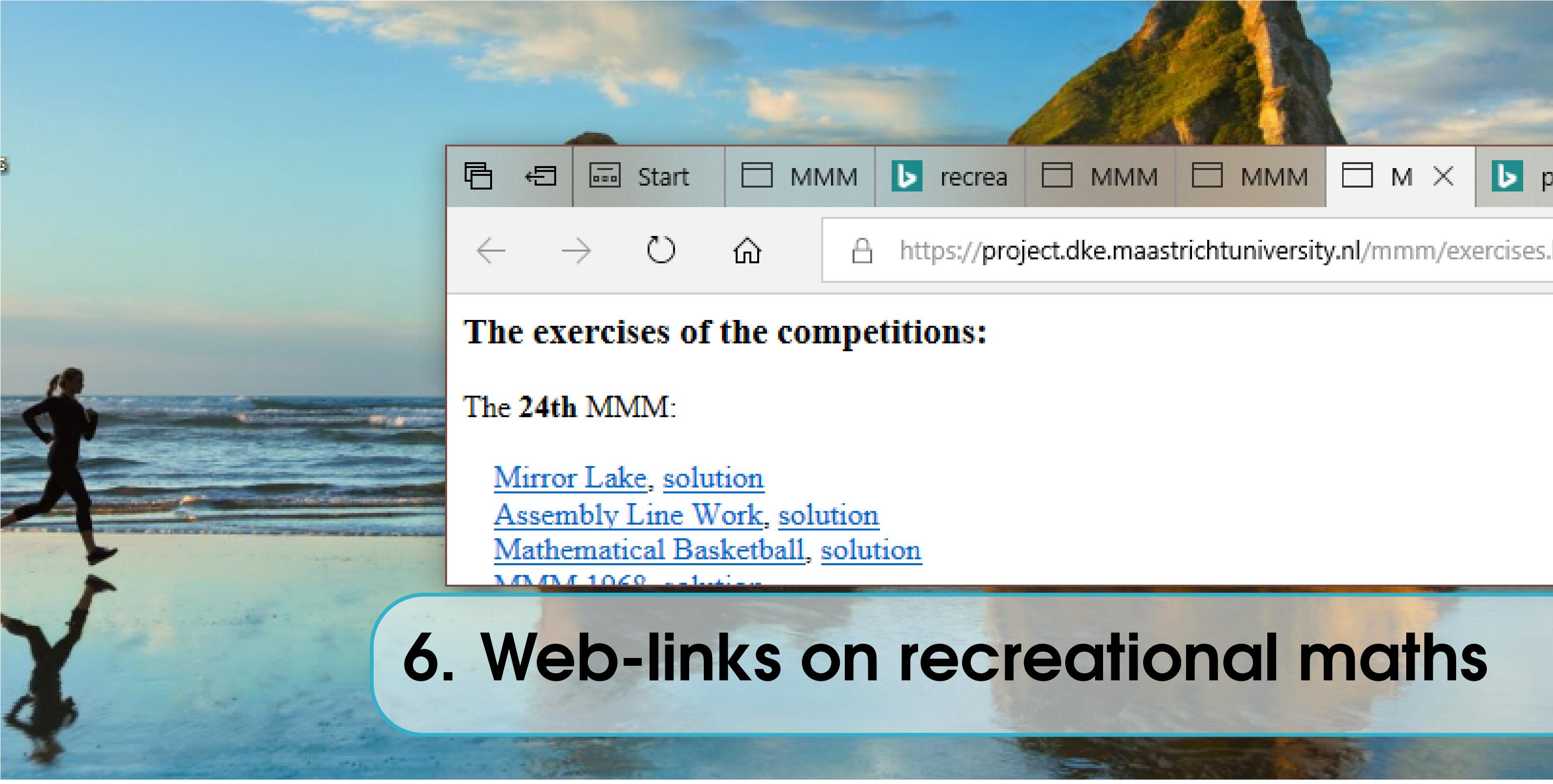
This gives:

$$|\mathrm{BM}|^2 = 30 \cdot 34/16 + 30 \cdot 34/4 - (30 \cdot 34/4) cos(\pi/\sqrt{34}) = 99.87,$$

And therefore $|\mathrm{BM}| = \sqrt{99.87} = 9.99$ cm.

Because the ant first crawled 2.51 cm to get from the left ear hole to the point B, and because it mirrored its path to get from M to the right ear hole, the total distance it crawled equals $2 \cdot (9.99 + 2.51) = 25.00$ centimeters.

Without computing their actual values, which is greater, e^{π} or π^{e}?



6. Web-links on recreational maths

Many of the MMM exercises are inspired on mathematical puzzles found on the internet, some of them dating back to the mid 19th century when Sam Lloyd started to publish his conundrums in American newspapers on a regular basis. Other famous problem inventors are Lewis Carroll, Martin Gardner and the Dutch journalist Léon Vié. When searching on the internet for interesting problems best thing to do is use the key 'recreational math problems'. You will be rewarded with lots of hits; below the best ones are displayed.

- https://mathschallenge.net/links/problems
- https://www.ocf.berkeley.edu/~ wwu/riddles/intro.shtml. This website provided the riddles underneath the chapters in this booklet. Solutions may be found over there as well.
- https://puzzle.dse.nl/index_us.htmlhttps://puzzle.dse.nl/index_us.html
- https://library.ucalgary.ca/c.php?g=255446&p=1703608

And naturally our own website:

- https://project.dke.maastrichtuniversity.nl/mmm/exercises.html

Searching for specific problems? The MMM exercises are categorized; see the table below.

E.	Year	1	2	3	4	5	Categories	
1	1995	P	C	L	P	R	**G**	Geometry
2	1996	C	C	A	R	P	**C**	Combinatorics
3	1997	C	R	L	G	G	**A**	Analysis
4	1998	C	A	A	L	L	**R**	Reasoning
5	1999	A	G	R	R	C	**L**	puzzLe
6	2000	G	R	N	A	A	**P**	Probability
7	2001	L	L	R	P	C	**N**	Numbers, integer equations
8	2002	A	L	R	G	N		
9	2003	C	P	L	L	G		
10	2004	P	A	A	C	A		
11	2005	G	C	C	G	N		
12	2006	A	R	N	N	R		
13	2007	R	A	C	P	A		
14	2008	A	R	N	L	N		
15	2009	R	R	R	G	N		
16	2010	P	A	A	R	R		
17	2011	G	R	P	R	A		
18	2012	R	N	R	L	N		
19	2013	L	A	A	R	A		
20	2014	A	N	A	R	A		
21	2015	R	R	P	R	N		
22	2016	A	R	A	C	A		
23	2017	P	R	A	A	A		
24	2018	G	A	N	N	R		
25	2019	G	L	C	R	A		

I have two coins. One of the coin is a faulty coin having tail on both sides. The other coin is a perfect one (heads on one side and tail on the other).
I blindfold myself and pick a coin and put it on the table. The face of the coin towards the sky is tail. What is the probability that the other side is also tail ?

Maastricht University

School of Business and Economics
and
Faculty of Science and Engineering

Certificate

handed out to the participant

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of the 25th *M*athematical *M*odelling competition *M*aastricht.

Maastricht,
January 26, 2019

Professor Dr. H. Peters	Dr. G. Schoenmakers
Department of Quantitative Economics	Department of Data Science and Knowledge Engineering

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