MMM Competition 2023

Problem 1.

There are 20 cookies on the table. You and your brother play the following game to eat the cookies. The game is played alternatingly. That means, you start, then your brother plays, then you play and so on. In every round, the person that plays eats 1, 2, or 3 cookies from the table. The person that eats the last cookie looses. A strategy is called *winning* if no matter how your opponent plays you can always guarantee to win the game.

(a) Who has a winning strategy: you or your brother?

(b) Explain the winning strategy.

(c) Suppose that instead of 20 cookies there are n cookies on the table. For which values of n do you, as a starting player, have a winning strategy?

Problem 2.

A number is called a perfect square if it is the square of some number. For example, 25 is a perfect square because $25 = 5^2$. Order the numbers from 1 to 15 in such a way that the sum of any two adjacent numbers is a perfect square.

(a) Given this order, what is the sum of the first and last number?

Now, suppose that instead numbers from 1 to 15, we use the numbers from 1 to n.

(b) What is the smallest value of n, with $n \geq 15$, for which no order exists such that the sum of any two adjacent numbers is a perfect square?

Problem 3.

Bert has a collection of 7 identical red marbles and a large number of identical blue marbles. He decides to arrange his 7 red marbles and some of the blue marbles in a row and he notices that the number of marbles whose right hand neighbor is the same color as themselves, is equal to the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is $BRRBBRRBBRRRB$ (notice that 6 marbles have a right hand neighbor of the same color, 6 marbles have a right hand neighbor of a different color and one marbles has no right hand neighbor). Let m be the *maximum* number of blue marbles for which such an arrangement is possible.

(a) Find m.

Now that we know m, let N be the number of arrangements of these $m + 7$ marbles that satisfy the requirement that 'the number of marbles whose right hand neighbor is the same color as themselves, is equal to the number of marbles whose right hand neighbor is the other color'.

(b) Find N.

Problem 4.

Consider the following sequence: $1001, 1004, 1009, 1016, 1025, \ldots$. The dots at the end men that this sequence goes on indefinitely, so the sequence has an infinite number of terms. Let a_n be the n^{th} term in this sequence. (So $a_1 = 1001, a_2 = 1004$ etc.)

(a) Find an expression of a_n in terms of n.

We are interested in finding the greatest common divisors (GCD) of consecutive terms a_n and a_{n+1} in the sequence. Notice that $GCD(1001, 1004) = 1$, as 1001 and 1004 do not share any prime factors. However, later in the sequence there might be consecutive terms a_n and a_{n+1} that have a GCD bigger than 1. Your task is to find the *maximum* value that the greatest common divisor of a_n and a_{n+1} can be. Or, in mathematical terms:

(b) Find $\max\limits_{n} GCD(a_n, a_{n+1})$.

Problem 5.

You gamble! Below are five multiple choice questions on optimality. You don't need to provide any computation or argumentation.

(a) If you connect locations in a flat country like the Netherlands with a railroad network of minimal total length, then any three railroad tracks meeting somewhere in the network always make angles of 120 degrees.
Sketched are three different ways of connecting 6 places located in the corners of a regular hexagon by a network that meets this optimality criterion.

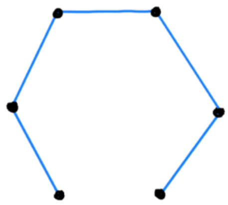


Figure 1

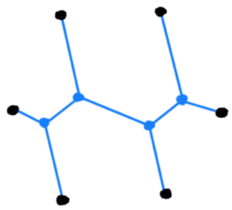


Figure 2

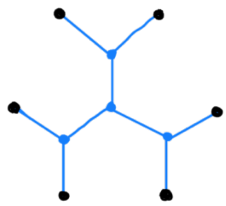
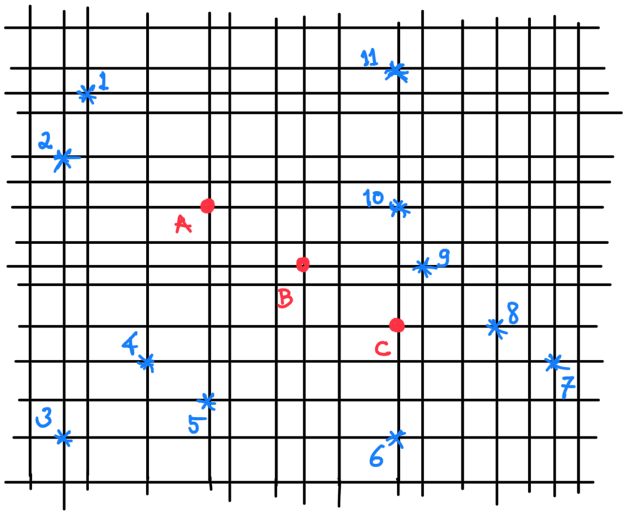


Figure 3

Which one has **minimal total length**?
(i) Figure 1
(ii) Figure 2
(iii) Figure 3

(b) Today, I was a bit clumsy, and I dropped a square glass mirror which broke into several sharp pieces when it hit the ground. Then I noticed that all the pieces happened to be triangles with acute angles only (so, all angles were strictly less than 90 degrees).
Into **how many pieces** did the mirror break **at least**?
(i) 6
(ii) 7
(iii) 8
(iv) 9

(c) The Council of Eleven (de Raod vaan èlf) of the Maastricht Carnival Association "De Tempeleers" went on a business trip to New York. They all booked different hotels in a part of Manhattan where the roads form a rectangular grid of Streets and Avenues. Below is a map of their locations. They would like to meet at a convenient location in town, such that **the total distance** they jointly must walk to get there is **minimal**.



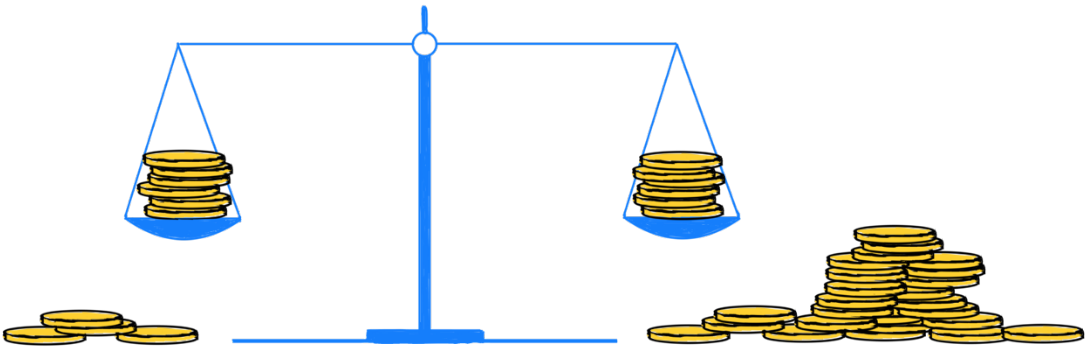
What is the best location for them to meet?

(i) Location A

(ii) Location B

(iii) Location C

(d) A fake gold coin got mixed up in a pile of 1000 real ones. All the 1001 coins look alike; from their visual appearance there is no telling which is real and which is fake. But the real coins all have the same weight, whereas the fake coin does not, though we don't know for sure if it is lighter or heavier.
At our disposal we have a traditional balance, by which we can compare the weight of two piles of coins placed at each of its scales, but which doesn't allow us to measure a numerical value.



What is the **least number of weightings needed** to always find the fake coin and determine whether it is lighter or heavier than the real coins?

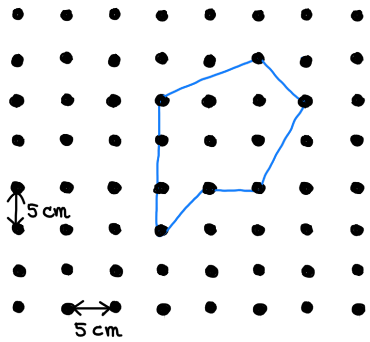
(i) 6

(ii) 7

(iii) 8

(iv) 9

(e) A fakir is playing with an elastic band on his bed of nails. These nails are arranged in a rectangular grid, with gaps of 5 cm horizontally and vertically between them. The fakir spans the elastic band around some of them. See the example below.



What is **the smallest area** that will always be enclosed when there are precisely 4 nails in the strict interior of the band (so, not counting those in the corners or along any edge)?

(i) 100 cm^2

(ii) 112.5 cm^2

(iii) 117.5 cm^2

(iv) 125 cm^2