Problem 1: Down to the wire

When visiting a friend, Sonja notices the following piece of art hanging on the wall.

She is much interested in the patterns and curves that are generated by the white pieces of string that are visible in the left bottom corner and the right upper corner.
To describe these curves, we introduce coordinates x (horizontal) and y (vertical), such that the four corner points are $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, respectively, with $(0,0)$ being the point in the left bottom corner. For convenience, we will only address the right upper part of the figure. Each of the white lines up there has the following property: it connects a point with x-coordinate $p \in [0,1]$ on the horizontal top edge of the figure with another point with y-coordinate $1-p$ on the vertical right edge. Sonja, then, is interested in the curve C to which these white lines are all tangent.

a (**1 point**) Specify three different points, with exact rational coordinates, that are on this curve C.

b (**2 points**) .] If one white line passes through the point $(p,1)$ and another white line through the point $(q,1)$, for two different values of p and q in the interval $[0,1]$, what are the coordinates of the intersection point of the two white lines?

c (**2 points**) Is the curve C part of a circle, of an ellipse (but not a circle), of a parabola, of a hyperbola, or of something else? [Choose precisely one; you may guess!]

d (**5 points**) Give a formula of the form $y = f(x)$ which describes the curve.

Problem 2: Two Circles

Consider two circles C1 and C2 in the plane, with radius 1 cm and 3 cm, and distance 10 cm between the two centers. What is the set of points that are the midpoint of a segment with one endpoint point on C1 and the other endpoint on C2?

Problem 3: Star Wars

Fred collects laminated pictures of Star wars characters. Those pictures are sold individually packed in opaque pouches and unknown in advance by the buyers, for a price of 1 Euro a pouch. (Hence, each pouch contains exactly one picture.) The seller has filled one third of the pouches with a picture of Rey, one fourth with a picture of Finn, one sixth with a picture of Poe, and the rest with pictures of some other characters. Knowing that Fred does not like to exchange his pictures with other people, and assuming the seller has produced infinitely many pictures, what is the average amount of money Fred should spend in order to get a picture of each of the three main Star wars characters?

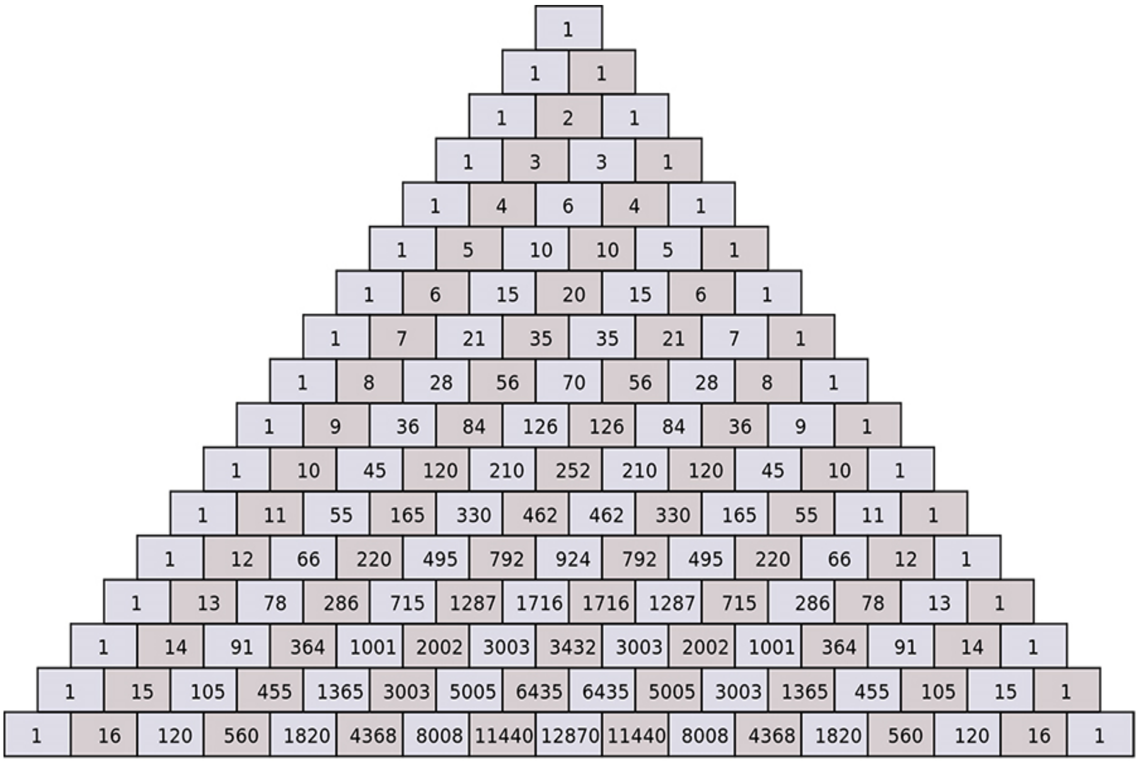
Problem 4: Loopy Logic

Below twenty statements are given. Which ones are true and which ones are false?

1) The answer to this statement is different from the answer to 20.
2) At least half of the statements in this puzzle are false.
3) At least half of the statements in this puzzle are true.
4) The answers to 8 and 13 are the same.
5) The answers to 12 and 16 are the same.
6) The answers to 19 and 20 are the same.
7) The answers to 8 and 9 are different.
8) The answers to 7 and 9 are different.
9) The answers to 7 and 8 are different.
10) 19 is true.
11) 16 is true.
12) 17 is true.
13) 14 is true.
14) 18 is false.
15) 3 is true.
16) 2 is true.
17) 5 is false.
18) 15 is true.
19) Prof. Peters' official first name is Hendrikus.
20) Prof. Peters has been working at Maastricht University since 1988.

Problem 5: Pascal's Revenge

Pascal's triangle (see picture below for the first several rows) is a famous mathematical structure named after the French mathematician Blaise Pascal. This triangle has lots of hidden (and not so well hidden) patterns. The most obvious pattern is of course that each number is equal to the sum of the two



numbers directly above it. Another well-known property is that the number in row n and column k is equal to $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the number of possible subsets of k items taken from a set of n items. Here the row and column numbering start at 0 and a column is going down diagonally rather than vertically. So the 1 at the top is in row 0 and column 0 (and it is of course well-known that $\binom{0}{0} = 1$). It turns out that in several rows of Pascal's triangle, there are three consecutive numbers that are in ratio $x : x+1 : x+2$. For instance in row 7 we find the numbers 7, 21 and 35 in columns 1, 2 and 3. These numbers are in ratio $1 : 3 : 5$ or $\frac{1}{2} : 1\frac{1}{2} : 2\frac{1}{2}$ (so $x = \frac{1}{2}$). Also in row 14 we have the beautiful sequence 1001, 2002 and 3003 starting in column 4 (ratio $1 : 2 : 3$). There are infinitely many of these in Pascal's triangle. Your task is to locate the highest numbered row before the 100th row with three consecutive numbers in it that are in ratio $x : x+1 : x+2$, as well as to determine x and the three corresponding columns.