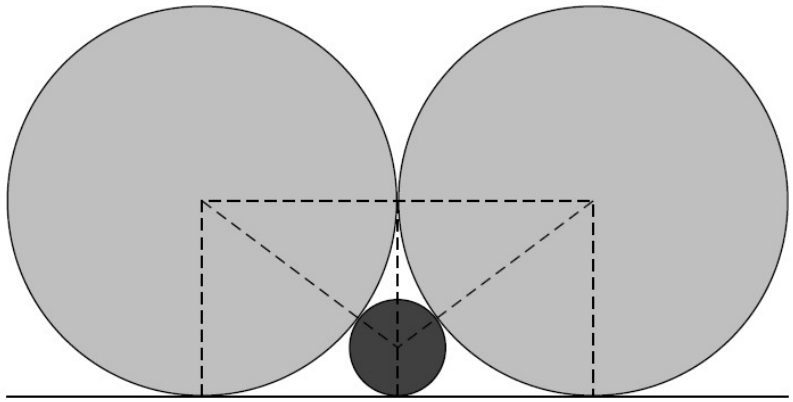
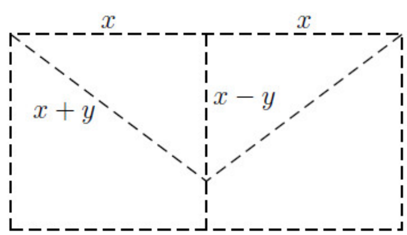




Answer to Problem 1: Three Circles

Here, x is the radius of the big circle. Since $\pi x^2 = 25$, $x = 5/\sqrt{\pi}$.
The radius of the small circle is y. By the Theorem of Pythagoras, $(x+y)^2 = x^2 + (x-y)^2$, hence $x^2 = 4xy$, so that $y = x/4 = 5/(4\sqrt{\pi})$. Therefore, the area of the dark gray disc is $\pi y^2 = 25/16 = 1.5625$.

Answer to Problem 2: Weight Watching

This is possible, as follows.

$d = 0$: $0, 0, 0, 0, 0$
$d = 1$: $1, 0, 0, 0, 0$
$d = 2$: $2, 0, 0, 0, 0$
$d = 3$: $3, 0, 0, 0, 0$
$d = 4$: $4, 0, 0, 0, 0$

$d = 5$: $1, 1, 1, 1, 1$
$d = 6$: $2, 1, 1, 1, 1$
$d = 7$: $3, 1, 1, 1, 1$
$d = 8$: $4, 1, 1, 1, 1$
$d = 9$: $5, 1, 1, 1, 1$

and in general for $d = 5x + y$, with $x = 0, 1, \ldots$ and $y = 0, 1, 2, 3, 4$:

$d = 5x + 0$: $x + 0, x, x, x, x$
$d = 5x + 1$: $x + 1, x, x, x, x$
$d = 5x + 2$: $x + 2, x, x, x, x$
$d = 5x + 3$: $x + 3, x, x, x, x$
$d = 5x + 4$: $x + 4, x, x, x, x$

This way we use 21 balls of weight 0, 22 balls of weight 1, 23 balls of weight 2, 24 balls of weight 3, and 25 balls of weight 4 or larger. To see this, note that we use balls of weight $x \geq 4$ on days $d = 5x + 0, 5x + 1, 5x + 2, 5x + 3, 5x + 4$ (in total 21 balls), and on days $5x - 4, 5x - 8, 5x - 12, 5x - 16$ (in total 4).

Answer to Problem 3: Line Dancing

a. A dance consists of 25 steps of which 14 are to the left. So we select 14 from the 25 steps that are going to be the steps to the left. This can be done in $\binom{25}{14}$ $(= 4457400)$ ways.

b. After the step to the right we still have 24 steps to go, 14 of which should be to the left. So $\binom{24}{14}$ $(= 1961256)$ dances start with a step to the right.

c. Notice that we can write a dance as a sequence of L and R of length 25, consisting of 14 Ls and 11 Rs. We have three different types of dances: the ones that start with an R, the ones that start with an L and that do return to the initial position some time during the dance and the ones that don't return to the initial position during the dance.

We need the following observation. Suppose that we have a dance d that starts with an R. Then after the first step prof. K has taken more steps to the right that to the left. Since at the end of the dance he has taken more steps to the left, we know that after some number of steps, say $p \in \{2, 4, \ldots, 22\}$ we have that for the last time he has taken exactly the same number of steps to the left as to the right.

Now consider dance $\hat{d}$, that is the 'complement' of dance d in the sense that at all steps until step p it takes a step in the opposite direction of dance d and after step p the steps are in the same direction as dance d. An example (for an eight-step dance): If $d = RRLLLRLL$, then $p = 6$, so the first 6 steps of $\hat{d}$ must be the opposite of d. So $\hat{d} = LLRRRLLL$. We can do this for every dance d that starts with an R, which shows that the number of dances starting with an R is exactly the same as the number of dances starting with an L that does return to the initial position some time during the dance.

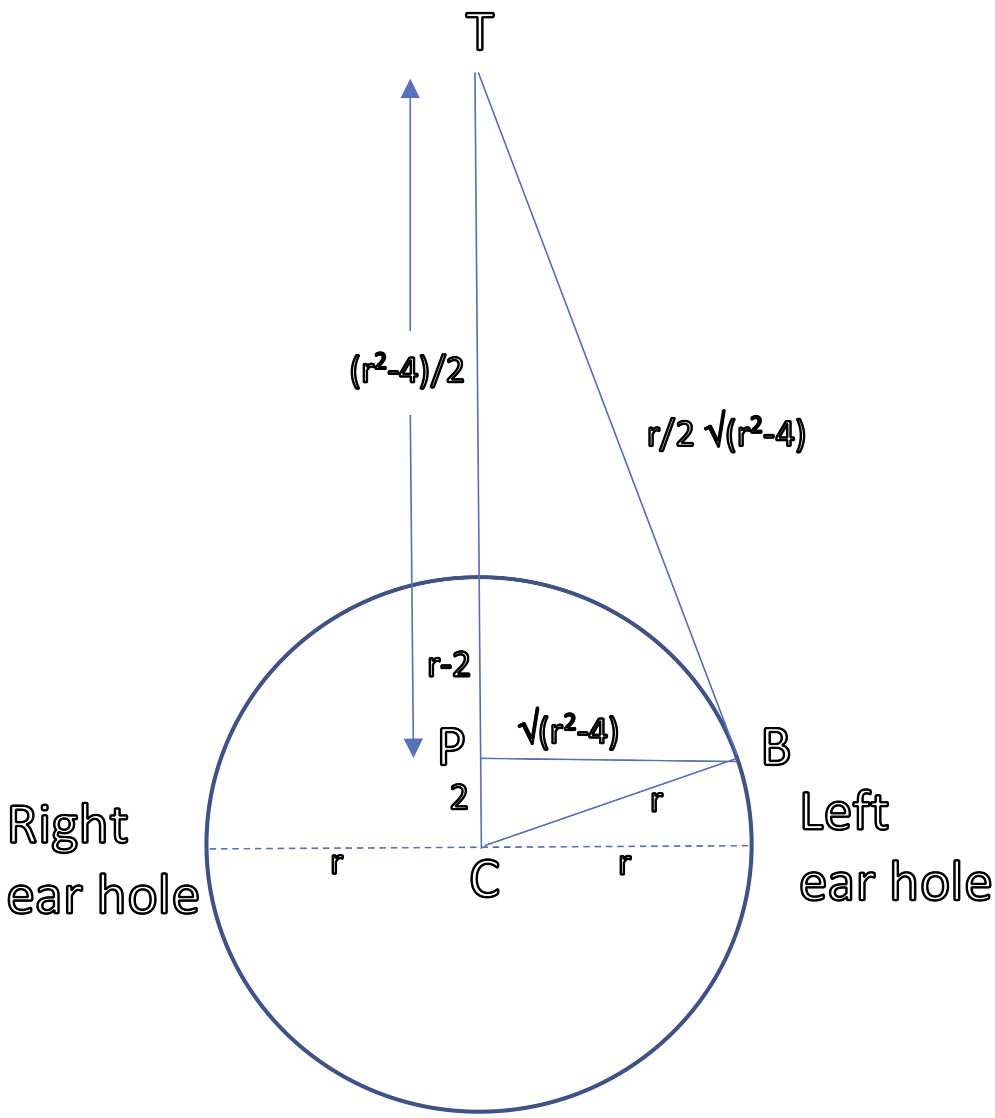
Therefore, let X be the number of dances that start with L and that do not return to its initial position during the dance. Then X is equal to the total number of dances minus two times the number of dances that start with an R. Or: $X = \binom{25}{14} - 2 \cdot \binom{24}{14} = 4457400 - 2 \cdot 1961256 = 534888$.

Answer to Problem 4: Cards in Space

Remark that Lando cannot win the first card, because the numbers between 1 and 24 are not divisible by 25. After the first card has been played, the remainder of the division by 25 of the sum of the cards on the table is a number i between 1 and 24. Only the card $24 - i$ can win at the next choice of Han (but maybe Han does not have it). Then, for each card he has played, Han knows the card Lando needs to have to win at the next turn. Moreover, Han knows the cards of Lando (all the numbers between 1 and 24 except the cards he had at the beginning and the cards they played). As Lando started Han has one more card than Lando when he plays, so he can choose a card such that Lando cannot win the next turn. Hence, Lando cannot win. Then, either Han wins before he plays his last card, or he plays his last card, and the sum $1 + 2 + ... + 24 = 24 * 25/2$ is divisible by 25. So Han wins all the time, and the probability is 1.

Answer to Problem 5: ‘Goes out one ear and into the other’

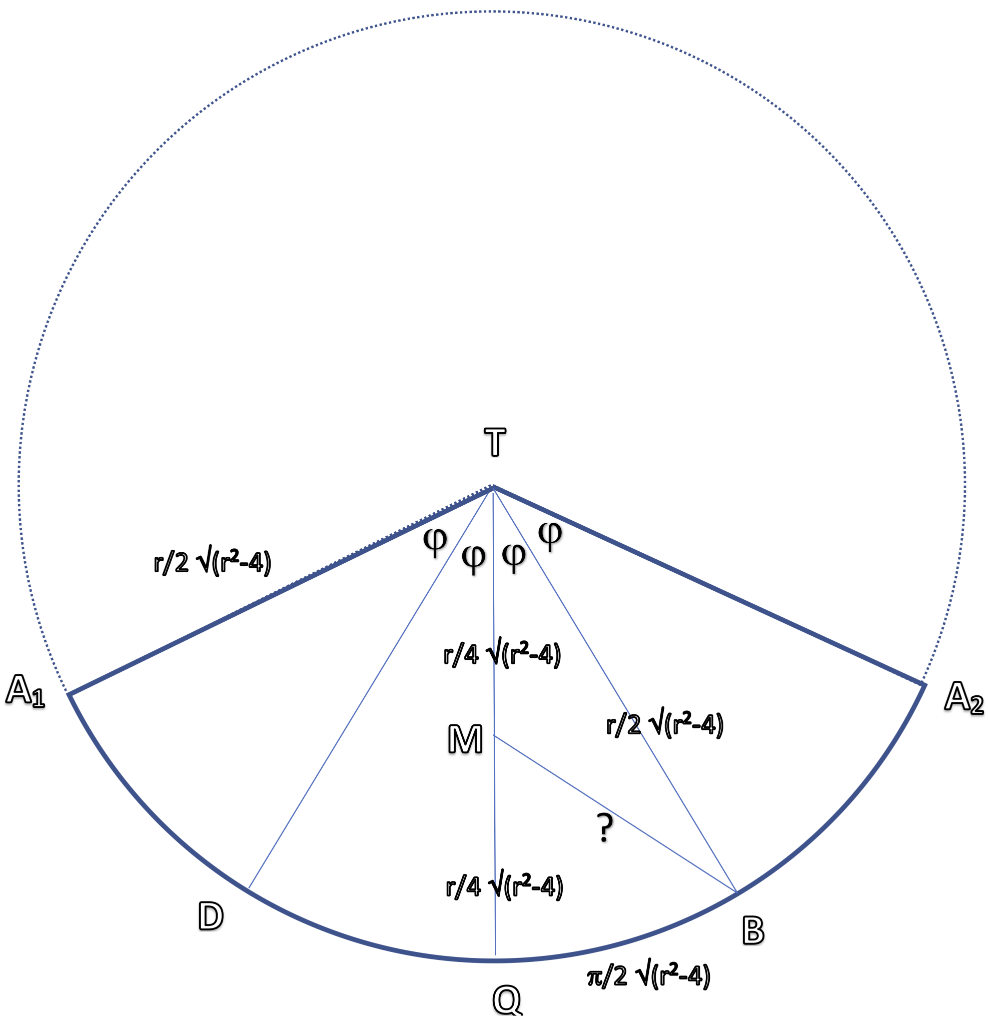
Question (a): When viewed from the front, the situation is as follows.



Working in centimeters, the diameter of the sphere equals $\sqrt{136}$, so that $r = \sqrt{34}$. From the center point C to the point B where the brim of the party hat touches the head of the garden gnome right above the left ear, the distance is also equal to r. Since B is in a horizontal plane that is 2 cm above the horizontal plane passing through the two ear holes, the center point P of the circle that constitutes the entire brim of the party hat, is located 2 cm above the point C. The angle BPC is a right angle, so according to Pythagoras’ law the radius of this circular brim of the party hat equals $\sqrt{r^2 - 4} = \sqrt{30}$.

The angle TBC is a right angle, because the party hat fits precisely, so it is tangent to the spherical head at the point B. Therefore, we have that the three triangles TBC, BPC, and TPB are all of the same shape (congruent), which allows us to calculate the distance |TB| as $(\sqrt{34}\sqrt{30})/2$ cm and the distance |TP| as $(r^2-4)/2 = 15$ cm. Finally, it follows that the distance |TC| equals $2+(r^2-4)/2 = 17$ cm, so that the top of the party hat is $17-\sqrt{34}$ cm above the top of the head of my garden gnome. Converting this to inches shows that my garden gnome with party hat is $(17-\sqrt{34})/2.54 = 4.40$ inches taller than 20.60 inches, so the answer is 25.00 inches.

Question(b): To determine the shortest path on the party hat, you should realize that a party hat is made every day by children in this world, cutting it from a flat piece of paper and bending it together. If you were to take a pair of scissors to cut it open at the back, the paper hat would simply flatten and take the shape displayed in the next figure. The sides TA_1 and TA_2 indicate the place where you used the



scissors; the point M indicates the center point of the five-pointed star of Maastricht (in the middle!) and the ant is located above the left ear, precisely at the point B on the brim of the hat. On this flat piece of paper you need to compute the length of BM. The points $A_1 = A_2$, B, Q, and D, are 4 points that divide the brim of the party hat into 4 equal pieces; Q is right in front. The total length of the brim follows from its radius before cutting the party hat open (see question (a)): it equals $2\pi\sqrt{r^2-4} = 2\pi\sqrt{30}$. The total length of the full circle with radius $\frac{r}{2}\sqrt{r^2-4}$ from which the flattened paper hat is just a sector, equals $\pi r\sqrt{r^2-4}$, which is $r/2$ times larger. So: 2π (the angle of a full circle) is $r/2$ times larger than 4φ (the angle of the sector) as the circumferences are proportional to the angles. This means that the top angle 4φ of the sector is $4\pi/r$ radians, which gives: $\varphi = \pi/r = \pi/\sqrt{34}$ radians.

To compute |BM| we now consider the triangle TBM and we use the "law of cosines" (which generalizes Pythagoras' law from right triangles to arbitrary triangles):

$$|\mathrm{BM}|^2 = |\mathrm{TM}|^2 + |\mathrm{TB}|^2 - 2|\mathrm{TM}| \cdot |\mathrm{TB}| cos(\varphi).$$

This gives:

$$|\mathrm{BM}|^2 = 30 \cdot 34/16 + 30 \cdot 34/4 - (30 \cdot 34/4) cos(\pi/\sqrt{34}) = 99.87,$$

And therefore $|\mathrm{BM}| = \sqrt{99.87} = 9.99$ cm.

Because the ant first crawled 2.51 cm to get from the left ear hole to the point B, and because it mirrored its path to get from M to the right ear hole, the total distance it crawled equals $2 \cdot (9.99 + 2.51) = 25.00$ centimeters.