# MMM Competition 2024

**Problem 1.**

There are 123 paintings lined up in an art center, numbered from 1 to 123. Each painting is initially colored white. Outside the center, there are 123 artists waiting to paint. First, artist 1 enters the art center and changes the color of every painting from white to black. Then artist 2 enters and changes the color of every second painting (paintings 2, 4, 6, . . . ) from black to white. Then artist 3 enters and changes the color (from white to black, or from black to white depending on the current color of the painting) of every third painting (3, 6, 9, . . . ). This then continues so that each artist $n$ changes the color of every $n$th painting.

(a) What are the final colors of paintings 10, 64 and 123?

(b) Out of the 123 paintings, how many paintings have a final color that is black?

Assume that we again start with 123 paintings that are initially colored white. Now, however, only the even numbered artists go in and change colors in the same way as described above.

(c) What is the final color of paintings 98 and 120?

**Problem 2.**

A real number is called rational if it can be expressed as the ratio of two integers, and irrational if it cannot be expressed as the ratio of two integers. An example of a rational number is $-\frac{1}{2}$ and of an irrational number is $\sqrt{2}$.

Every irrational number can be written as an infinite expression of the form: $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$, where $a_0, a_1, a_2, a_3, \ldots$ are integers.

(a) Prove that $\sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$.

(b) Find values for $a_0$, $a_1$, $a_2$, $a_3$, $\ldots$ such that $\sqrt{2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$.

Alice, Bob and Charly all have either black or blond hair. Alice can only see Bob. Bob can only see Charly. Alice has black hair and Charly has blond hair.

(c) Can a person with black hair see a person with blond hair? Choose one of the following three answers: yes, no, or we don't know.

Using the same reasoning as in (c) and the fact that $\sqrt{2}$ is irrational, argue why the following statement is true.

(d) There exists two irrational numbers $x$ and $y$ such that $x^y$ is rational.

**Problem 3.**

After defeating El Supremo, Freddy the Frog (yes, the one from the 'well known' movie from 1992) is happily jumping around the coordinate plane, looking for the river, which lies on the horizontal line $y = 24$. There is also a fence, which is located at the horizontal line $y = 0$. Since Freddy is kind of drunk, he can't remember where the river is and on each jump he randomly chooses a direction parallel to one of the coordinate axes and jumps one unit in that direction. Whenever he is at a point where $y = 0$ (so at the fence), with equal probabilities he chooses one of three directions where he either jumps parallel to the fence or jumps away from the fence, but he never chooses the direction that would have him cross over the fence to where $y < 0$. Freddy starts his search at the point $(0, 21)$ and will stop once he reaches a point on the river. Find the expected number of jumps it will take Freddy to reach the river.

**Problem 4.**

Initially Alice, Bob, and Charlie have a lot of peanuts: Between them they have a total of 1397 peanuts, where Charlie had the most peanuts and Alice the least. It turns out that the three numbers of peanuts that each person has, form a geometric progression. A geometric progression is a sequence of numbers in which the ratio between any two consecutive numbers is a constant. The three of them are getting hungry from looking at their peanuts, so they eat some of their peanuts. To be precise: Alice eats 5 of her peanuts, Bob eats 9 of his peanuts, and Charlie eats 24 of his peanuts. And now something peculiar has happened: The three numbers of peanuts that each person now form an arithmetic progression. An arithmetic progression is a sequence of numbers in which the difference between any two consecutive numbers is a constant.

Find the number of peanuts Alice had initially, as well as the common ratio in the geometric progression.

**Problem 5.**

Jack Bauer writes down the natural numbers from 1 to 24 in decreasing order:

$$24 \ 23 \ 22 \ 21 \ 20 \ 19 \ 18 \ 17 \ 16 \ 15 \ 14 \ 13 \ 12 \ 11 \ 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1$$

You are asked to first remove as many of these numbers as you want, and then to insert between them the operations subtraction and addition, alternatingly, starting with subtraction. This leaves you with an expression to calculate. (Should you decide to remove all the numbers, the outcome is defined to equal 0.)

Example: you may choose to first remove the numbers 23, 22, 19, 16, 12, 11, 10, 9, 5, 4, 2, 1, and then insert the operations subtraction and addition as described, which leaves you with

$$24 - 21 + 20 - 18 + 17 - 15 + 14 - 13 + 8 - 7 + 6 - 3$$

If you calculate the outcome of this expression, you find the answer 12.

(a) What are all the possible outcomes of this procedure? (Explain.)

(b) How many different expressions can you create in this way? (Explain.)

(c) Following the rules above, in how many ways can you create an expression which equals 3? (Show all your work.)