Answer to Problem 1: Down to the Wire

(a) Two points are easy to see: $(0,1)$ and $(1,0)$. A third one is obtained from symmetry: for $p=\frac{1}{2}$ we get the line passing through $(\frac{1}{2},1)$ and $(1,\frac{1}{2})$. Its middle point is on the diagonal $x=y$ and located at $(\frac{3}{4},\frac{3}{4})$. This point is also on C.

(b) What is the description of a line passing through $(p,1)$ and $(1,1-p)$? In moving from $(p,1)$ to $(1,1-p)$ the horizontal translation is $1-p$ while the vertical translation equals $-p$. Therefore, the slope equals $\frac{-p}{1-p}$. It now easily follows that this line is described by:

$$y=-\tfrac{p}{1-p}x+\tfrac{p}{1-p}+1-p$$

For the line with parameter q we have likewise $y=-\frac{q}{1-q}x+\frac{q}{1-q}+1-q$. At the intersection point of these two lines we find that the same y-value occurs if

$$-\tfrac{p}{1-p}x+\tfrac{p}{1-p}+1-p=-\tfrac{q}{1-q}x+\tfrac{q}{1-q}+1-q$$

which is equivalent to

$$[\tfrac{q}{1-q}-\tfrac{p}{1-p}]x=\tfrac{q}{1-q}+1-q-[\tfrac{p}{1-p}+1-p]$$

or

$$[q(1-p)-p(1-q)]x=q(1-p)+(1-q)^2(1-p)-p(1-q)-(1-p)^2(1-q)$$

This can be rearranged as $(q-p)x=q-p+(1-p)(1-q)(p-q)$, which gives

$$x=1-(1-p)(1-q)=p+q-pq$$

Plugging this into either one of the two descriptions for the lines yields the corresponding y-value as $y=p(1-q)+1-p=1-pq$. Summarizing, the intersection point has coordinates: $(1-(1-p)(1-q),1-pq)$ or

$$(p+q-pq,1-pq)$$

(c) It's a parabola. See after part (d).

(d) Consider the intersection point for the two lines with parameters p and q, computed under (b). If we keep one line (with parameter p) fixed and we

vary the other parameter q, we see that their intersection point converges to the tangent point of the fixed line at C, if q converges to p. So, if we choose $p = q$, we find that the corresponding point is on C; this is the point $(2p - p^2, 1 - p^2)$.
Now, what we need to do is to express $y = 1 - p^2$ in terms of $x = 2p - p^2$. Clearly, $1 - x = 1 + p^2 - 2p = (1 - p)^2$, which gives $1 - p = \sqrt{1 - x}$, so that we can express p as:

$$p = 1 - \sqrt{1 - x}$$

It then follows that $y = 1-p^2 = 1-(1-\sqrt{1 - x})^2$. Expanding the right-hand side, the answer we were looking for is:

$$y = 2\sqrt{1 - x} - (1 - x)$$

Returning to question (c): note that $y + (1 - x) = 2\sqrt{1 - x}$. Squaring gives: $(y - x + 1)^2 = 4(1 - x)$. This produces the following expression for the curve C (with symmetric roles for x and y):

$$x2 + y2 - 2xy + 2x + 2y - 3 = 0$$

This can also be written as $(x - y)^2 + 2(x + y) - 3 = 0$. If we now introduce new coordinates u and v as: $u = x - y$, $v = x + y$ (which corresponds to a rotation by 45 degrees and scaling by a factor $\frac{1}{2}\sqrt{2}$, then $u^2 + 2v - 3 = 0$, or:

$$v = (-u2 + 3)/2$$

This shows that C is part of a parabola.

Answer to Problem 2: Two Circles

The answer is: all points with distance at least 1 and at most 2 from the midpoint of the line segment with the centers of C1 and C2 as endpoints. Hence, it is the area between the circles with radius 1 cm and 2 cm with this center. (Formally, this is called a torus.)

Answer to Problem 3: Star Wars

Let x_R denote the average number of cards one has to buy until a picture of Rey shows up. With probability 1/3, this happens immediately, and with probability 2/3, one has to start again. Hence, x_R is a solution of the equation

$$x_R = 1/3 \cdot 1 + 2/3 \cdot (1 + x_R)$$

which implies $x_R = 3$. Similarly, $x_F = 4$ and $x_P = 6$.
By similar logic, the expected number of cards one has to by in order to get pictures of Finn and Poe, x_{FP}, satisfies the equation

$$x_{FP} = 1/4 \cdot (1 + x_P) + 1/6 \cdot (1 + x_F) + 7/12 \cdot (1 + x_{FP})$$

which implies after a straightforward computation that $x_{FP} = 7\frac{3}{5}$. Similarly, $x_{RP} = 7$ and $x_{RF} = 5\frac{2}{7}$.
Finally, the desired number x_{RFP} satisfies the equation

$$x_{RFP} = 1/3 \cdot (1 + x_{FP}) + 1/4 \cdot (1 + x_{RP}) + 1/6 \cdot (1 + x_{RF}) + 1/4 \cdot (1 + x_{RFP})$$

with solution $x_{RFP} \approx 8.22$.

Answer to Problem 4: Loopy Logic

We know:

- If statement 20 is true, then we arrive at a contradiction concerning statement 1: If 1 is true, it must be false and if 1 is false it must be true. That's not possible, so **statement 20 is false**.
- 12 True $\Leftrightarrow$ 17 True $\Leftrightarrow$ 5 False. We conclude that

    1. 12 True $\Rightarrow$ 17 True $\Rightarrow$ 5 False $\Rightarrow$ 16 False
    2. 12 False $\Rightarrow$ 17 False $\Rightarrow$ 5 True $\Rightarrow$ 16 False

  so either way **statement 16 is false**

- From 16 being false we conclude that **statements 2 and 11 are both false**, and from 2 being false we conclude that **statement 3 is true**. From this we conclude that **statement 15 is true** and then that **statement 18 is true** and then that **statement 14 is false** and then that **statement 13 is false**.
- From 2 being false we also conclude that there are at most nine false statements. We already have found six false statements (2, 11, 13, 14, 16 and 20) We know that statements 7, 8 and 9 are either all false or one of them is false and the other two are true. Since one of statements 5 and 17 must be false, we conclude that **exactly one of the statements 7, 8 and 9 is false**.
- If 8 is true, then 4 is false and we have too many false statements, so **statement 8 is false**. And apparently **statements 4, 7 and 9 are all true**.
- At this moment we have seven false statements. Statements 10 and 19 have the same truth value and one of statements 5 and 17 must be false. This means that **statements 10 and 19 are both true**. But then **statement 6 is false.**
- Finally we now need that **statements 12 and 17 are true and statement 5 is false** (otherwise we have too many false statements) and we apparently also need that **statement 1 is true**.

Answer to Problem 5: Pascal's Revenge

For three consecutive numbers in row n and columns k, $k+1$ and $k+2$ to be in ratio $x : x+1 : x+2$ we need that

$$(x+1)(x+2)\binom{n}{k} = x(x+2)\binom{n}{k+1} = x(x+1)\binom{n}{k+2}$$

Solving the left equation for x yields $x = \frac{k+1}{n-2k-1}$; the right equation gives $x = \frac{-n+3k+5}{n-2k-3}$. Setting these two numbers equal and solving for n gives

$$n_{1,2} = \frac{4k+5 \pm \sqrt{8k+17}}{2}$$

We need n and k to be nonnegative integers. So if we let $m = 8k+17$, then m must be a natural number as well. It turns out that if we choose m equal to 5, 7, 9, 11, 13, ... (so an odd number greater than or equal to 5), then we always find an integer-valued solution for k (this can quite easily be proved). To get the increasing sequence (corresponding to the ratios $x : x+1 : x+2$ we then need to take $n = \frac{4k+5+\sqrt{8k+17}}{2}$. The value of x can then be found by solving the equation $x = \frac{k+1}{n-2k-1}$ (or the other equation). The following table gives the values for n and k for values of m up to 19:

m	k	n	x
5	1	7	$\frac{1}{2}$
7	4	14	1
9	8	23	$1\frac{1}{2}$
11	13	34	2
13	19	47	$2\frac{1}{2}$
15	26	62	3
17	34	79	$3\frac{1}{2}$
19	43	98	4

For $m = 21$ the corresponding value of n is going to exceed 100, so apparently the row number that is asked is row 98 and in columns 43, 44 and 45 there are three numbers that are in ratio $4 : 5 : 6$.
Extra: The numbers in this sequence are

$$\binom{98}{43} = 12\,289\,694\,242\,827\,235\,919\,344\,118\,592$$
$$\binom{98}{44} = 15\,362\,117\,803\,534\,044\,899\,180\,148\,240$$
$$\binom{98}{45} = 18\,434\,541\,364\,240\,853\,879\,016\,177\,888$$